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A Survey on the Arthur-Selberg Trace Formula *

Takuya KONNO †

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†Graduate School of Math., Kyushu University
812-8581, Hakozaki 6-10-1, Higashi-ku, Fukuoka city, Japan

1 Introduction

In this note, we present a brief survey on the Arthur-Selberg trace formula. Interested readers can consult more detailed expositions [1], [28], [29], and of course, the original papers [2] to [13]. See also [20] for some important ideas and several appropriate arguments in reduction theory. For the purpose of this introduction, it is sufficient to recall the original Selberg trace formula and give some words about arithmetic backgrounds.

The Selberg trace formula was originally proved for a pair $(\mathbf{G}, \Gamma)$ of a semisimple Lie group and a cocompact discrete subgroup in it [37], [38]. If we exclude some exceptional cases, this is equivalent to the setting of anisotropic adéle groups.

Thus let F be a number field and write $\mathbb{A} = \mathbb{A}_F$ for its ring of adéles. $| \, |_{\mathbb{A}}$ denotes the idéle norm on $\mathbb{A}^\times$ and set $\mathbb{A}^1 := \mathrm{Ker}| \, |_{\mathbb{A}}$. For a connected semisimple group G over F, its group of adelic points $G(\mathbb{A})$ is a locally compact topological group, in which the group $G(F)$ of F-rational points is a discrete subgroup. We assume that G is *anisotropic* over F, that is, $G(F)$ contains only semisimple elements. Then $G(F)\backslash G(\mathbb{A})$ is *compact* by the result of [18].

R denotes the right regular representation of $G(\mathbb{A})$ on the space $L^2(G(F)\backslash G(\mathbb{A}))$. Write $C_c^\infty(G(\mathbb{A}))$ for the space of functions with compact supports on $G(\mathbb{A})$ which is smooth in the archimedean components and locally constant in the non-archimedean components. For $f \in C_c^\infty(G(\mathbb{A}))$, the operator

$$\begin{aligned}[R(f)\phi](x) &:= \int_{G(\mathbb{A})} f(y)\phi(xy)\,dy = \int_{G(\mathbb{A})} f(x^{-1}y)\phi(y)\,dy \\ &= \int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\phi(y)\,dy\end{aligned}$$

is an integral operator with the kernel

$$K(x,y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

Since $G(F)\backslash G(\mathbb{A})$ is compact, this operator is of *Hilbert-Schmidt class* (i.e. $K(x,x)$ is square integrable on $G(F)\backslash G(\mathbb{A})$). In particular one can show that the representation R decomposes into a direct sum of irreducible representations where each irreducible representation occurs with finite multiplicity:

$$R = \bigoplus_{\pi \in \Pi(G(\mathbb{A}))} \pi^{\oplus m(\pi)} \tag{1.1}$$

Here, $\Pi(G(\mathbb{A}))$ denotes the set of isomorphism classes of irreducible unitary representations of $G(\mathbb{A})$. Moreover an argument of Duflo-Labesse [21, I.1.11] shows that $R(f)$ is of *trace class.* That is, it admits a trace given by the integral of $K(x,y)$ on the diagonal:

$$\begin{aligned}\mathrm{tr}R(f) &= \int_{G(F)\backslash G(\mathbb{A})} K(x,x)\,dx = \sum_{\{\gamma\}\in\mathfrak{O}(G)} \sum_{\delta\in G^{\gamma}(F)\backslash G(F)} \int_{G(F)\backslash G(\mathbb{A})} f(x^{-1}\delta^{-1}\gamma\delta x)\,dx \\ &= \sum_{\{\gamma\}\in\mathfrak{O}(G)} \frac{1}{[G^{\gamma}(F):G_{\gamma}(F)]} \int_{G_{\gamma}(\mathbb{A})\backslash G(\mathbb{A})} \int_{G_{\gamma}(F)\backslash G_{\gamma}(\mathbb{A})} f(x^{-1}\gamma x)\,dy\,dx \\ &= \sum_{\{\gamma\}\in\mathfrak{O}(G)} a^{G}(\gamma) I_G(\gamma,f).\end{aligned}$$

Here $\mathfrak{O}(G)$ is the set of (semisimple) conjugacy classes in $G(F)$, $G^{\gamma} := \mathrm{Cent}(\gamma, G)$ is the centralizer of γ in G, $G_{\gamma} := \mathrm{Cent}(\gamma, G)^0$ is its identity component, and

$$a^{G}(\gamma) := \frac{\mathrm{meas}(G_{\gamma}(F)\backslash G_{\gamma}(\mathbb{A}))}{[G^{\gamma}(F):G_{\gamma}(F)]}, \quad I_G(\gamma,f) := \int_{G_{\gamma}(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\,dx.$$

This combined with (1.1) yields the *Selberg trace formula*:

$$\sum_{\{\gamma\}\in\mathfrak{O}(G)} a^{G}(\gamma) I_G(\gamma,f) = \sum_{\pi\in\Pi(G(\mathbb{A}))} a^{G}(\pi) I_G(\pi,f), \tag{1.2}$$

where $a^G(\pi) := m(\pi)$ and $I_G(\pi, f) := \mathrm{tr}\pi(f)$ is the distribution character of π. The left and right sides are called the *geometric* and the *spectral* side, respectively.

The Selberg trace formula has many variations reflecting its applications to wide variety of fields in mathematics. But the Arthur-Selberg trace formula is the only extension to the case when G has the F-rank greater than one. It is one of the important ingredients of the Langlands program. Perhaps it will be helpful to explain the program briefly.

It is a program to understand the deep relationship between automorphic forms and Galois representations and motives. Two main processes are

(1) To describe the automorphic representations of reductive groups by means of their associated automorphic L-functions (or their Langlands parameters);

(2) To construct the correspondence between automorphic representations of arithmetic type and ℓ-adic representations of Galois groups of the field of definition of such forms. This correspondence should be characterized by the equality between automorphic and Artin L-functions.

Both of them are very hard but will provide a lot of fruitful arithmetic informations.

As for (1), the case of inner forms of $GL(n)$ was successfully treated [25], [26], [35] and [15]. The relevant L-functions are the Rankin product L-functions. Then one expects to reduce the case of general reductive (or at least classical) groups to the $GL(n)$ case. This divides into two steps. First we relate the automorphic representations of a reductive group G to those of its quasisplit inner form G^*. This is suggested by the experience in the $GL(n)$ case [24], [15]. Then relate the automorphic representation of G^* with those of some $GL(n)$. In [14, § 8] the detailed framework of the second part for classical G are explained. We need to compare the trace formula for G with that of G^* in the first problem, while a *twisted* trace formula for $GL(n)$ need to be compared with the ordinary one for G^* in the second.

The problem (2) contains so many aspects that we cannot explain them in any detail here. But the starting point is to construct the Galois representations associated to an arithmetic automorphic representations in the ℓ-adic cohomology of certain Shimura variety. Here the trace formula with special geometric test functions will be compared with the Lefschetz-Verdier trace formula of the Shimura variety.

For these purposes, the trace formula must be of the arithmetic form, i.e. must be *stabilized*. In fact, already in the spectral decomposition of the L^2-automorphic spectrum, the normalization of intertwining operators by L-functions and the precise analytic properties of those L-functions must be established. The analytic trace formula alone yields little arithmetic information !

In this note, however, we deal only with the analytic aspects in the construction of the Arthur-Selberg trace formula. Some parts of the stabilization process will be found in the article of Hiraga in this volume. The contents of this note is as follows. We start with a brief review of Langlands' theory of spectral decomposition of the automorphic spectrum by Eisenstein series § 2. In the higher rank case, the residual Eisenstein series appears which makes the construction in what follows much more complicated. Anyway this allows us to express the kernel of the right translation operator in two forms, one geometric and the other spectral. § 3 explains the construction of [2] and [3]. We define the truncated kernel and prove that its integral over the diagonal converges. We obtain the coarse trace formula. § 4 is devoted to the fine $\mathfrak{O}$-expansion. We write the geometric terms in terms of the unipotent terms of certain reductive subgroups of G. Then they are expressed by means of weighted orbital integrals. The spectral counter part, the fine $\mathfrak{X}$-expansion, is explained in § 5. This is the heart of this note, because it is the most important part in applications. We use the fact that the coarse $\mathfrak{X}$-expansion is a polynomial in the truncation parameter T to deduce the precise expression of the $\mathfrak{X}$-expansion from the asymptotic formula of the inner product of truncated Eisenstein series. Finally in § 6, we illustrate the rough idea of Arthur's construction of the invariant trace formula [11], [12].

Because of the lack of time and volume, many important ideas are overlooked. In particular, we ignore many convergence/finiteness arguments, and also cannot refer to the works of Osborne-Warner [34]. Instead we add some examples in the simplest case $G = GL(2)$ with some figures.

2 Spectral decomposition

We begin with some notation. For a finite set of places S of F, we write $\mathbb{A}_S := \{(a_v)_v \in \mathbb{A} \mid a_v = 0,\ \forall v \notin S\}$. The infinite and finite components of $\mathbb{A}$ are denoted by $\mathbb{A}_\infty$ and $\mathbb{A}_f$, respectively.

Let G be a connected reductive group over F. For brevity, we write $\mathbf{G} := G(\mathbb{A})$, $\mathbf{G}_S := G(\mathbb{A}_S)$, etc. Let A_G be the maximal F-split torus in the center $Z(G)$ of G, while the maximal $\mathbb{R}$-vector subgroup in the center $Z(\mathbf{G}_\infty)$ of $\mathbf{G}_\infty$ is denoted by $\mathfrak{A}_G$. Write $\mathfrak{a}_G$ for its Lie algebra. We have a direct product decomposition $\mathbf{G} = \mathbf{G}^1 \times \mathfrak{A}_G$ such that

$$|\chi(ag)|_{\mathbb{A}} = |\chi(a)|_{\mathbb{A}}, \quad a \in \mathfrak{A}_G,\ g \in \mathbf{G}^1$$

holds for any F-rational character of G. Set

$$H_G : \mathbf{G} = \mathbf{G}^1 \times \mathfrak{A}_G \xrightarrow{\text{proj}} \mathfrak{A}_G \xrightarrow{\log} \mathfrak{a}_G.$$

For a parabolic subgroup $P = MU$, we write $\mathcal{F}(M)$, $\mathcal{P}(M)$ and $\mathcal{L}(M)$ for the set of parabolic subgroups containing M, the set of parabolic subgroups having M as a Levi factor and the set of Levi subgroups containing M. We fix a minimal parabolic subgroup $P_0 = M_0U_0$. $\mathcal{F}(P_0)$ and $\mathcal{L}(P_0)$ denotes the set of parabolic subgroups containing P_0 and the set of Levi components of elements of $\mathcal{F}(P_0)$ containing M_0 (the set of *standard* parabolic and Levi subgroups).

Fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ of $\mathbf{G}$ such that the Iwasawa decomposition $\mathbf{G} = \mathbf{PK}$ holds for any $P \in \mathcal{F}(M_0)$. Using this we extend the map $H_M : \mathbf{M} \to \mathfrak{a}_M$ to

$$H_P : \mathbf{G} = \mathbf{UMK} \ni umk \longmapsto H_M(m) \in \mathfrak{a}_M.$$

$W = W^G$ denotes the (relative) Weyl group $\mathrm{Norm}(A_0, G)/M_0$ of $A_0 = A_{M_0}$ in G, where $\mathrm{Norm}(H, G)$ means the normalizer of a subgroup H in a group G. We identify W with a fixed system of representatives in $\mathrm{Norm}(A_0, G)$. W acts by conjugation on $\mathcal{F}(M_0)$ and the associated objects. We write the action $w : P \mapsto w(P) = {}^wP := wPw^{-1}$ etc.

2.1 Siegel domains

Fixing an invariant measure on $\mathbf{G}$ and a Lebesgue measure on $\mathfrak{A}_G$, we can consider the space

$$L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G}) := \left\{ \begin{array}{c} \phi : \mathbf{G} \to \mathbb{C} \\ \text{measurable} \end{array} \middle| \begin{array}{ll} \text{(i)} & \phi(\gamma a g) = \phi(g),\ \forall a \in \mathfrak{A}_G,\ \gamma \in G(F) \\ \text{(ii)} & \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} |\phi(g)|^2\, dg < +\infty \end{array} \right\}$$

as in the anisotropic case. Let $C_c^\infty(\mathbf{G}/\mathfrak{A}_G)$ for the space of smooth functions on $\mathbf{G}$ which are compactly supported modulo $\mathfrak{A}_G$. Similar calculation as in the anisotropic case shows that, the operator

$$[R(f)\phi](x) := \int_{\mathfrak{A}_G\backslash\mathbf{G}} f(y)\phi(xy)\, dy$$

on $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ is an integral operator with the kernel

$$K(x, y) := \sum_{\gamma\in G(F)} f(x^{-1}\gamma y) \tag{2.1}$$

for $f \in C_c^\infty(\mathbf{G}/\mathfrak{A}_G)$. Later we shall use the subspace $\mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$ of elements which are $\mathbf{K}$-finite on both sides in $C_c^\infty(\mathbf{G}/\mathfrak{A}_G)$ as the space of test functions.

The spectral decomposition of the right regular representation $R = R_G$ of $\mathbf{G}$ on $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ is more complicated than in the anisotropic case, because $G(F)\mathfrak{A}_G\backslash\mathbf{G} = G(F)\backslash\mathbf{G}^1$ is no longer compact. The form of the non-compactness is described as follows.

Since M_0 is anisotropic modulo center by definition, we can choose a compact subset ω_1 of $\mathbf{M}_0^1$ satisfying $\mathbf{M}_0^1 = M_0(F)\omega_1$. As U_0 is a multiple extension of additive groups, there exists a compact subset $\omega_2 \subset \mathbf{U}_0$ such that $\mathbf{U}_0 = U_0(F)\omega_2$. For $T \in \mathfrak{a}_0$, we set

$$\mathfrak{A}_0(T) := \{a \in \mathfrak{A}_0 \mid \alpha(H_0(a) - T) > 0, \forall \alpha \in \Delta\},$$

where we have abbreviated $\mathfrak{A}_{M_0}$, H_{P_0} as $\mathfrak{A}_0$, H_0, respectively, and Δ is the set of simple roots of $A_0 := A_{M_0}$ in P_0, considered as a subset of $\mathfrak{a}_0^*$.

Proposition 2.1 ([23]). *We can choose $T_0 \in \mathfrak{a}_0$ sufficiently negative so that*

$$\mathbf{G} = G(F)\mathfrak{S}(T_0), \quad \mathfrak{S}(T_0) := \omega_2\omega_1\mathfrak{A}_0(T_0)\mathbf{K}$$

Thus the non-compactness comes from that of $\mathfrak{A}_0(T_0)$, a shifted positive cone.

2.2 Cusp forms

For an F-parabolic subgroup $P = MU$ of G and a measurable function ϕ on $U(F)\backslash\mathbf{G}$, we can define its constant term along P by

$$\phi_P(g) := \int_{U(F)\backslash\mathbf{U}} \phi(ug)\, du.$$

The space of *L^2-cusp forms* $L_0^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ consists of $\phi \in L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ such that ϕ_P vanishes almost everywhere for any $P \in \mathcal{F}(P_0)$. Of course this is not the space of cusp forms $\mathcal{A}_0(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ in the usual sense [33, Chapt. I], but $\mathcal{A}_0(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ is a dense subspace of $L_0^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$. The following lemma is fundamental in our estimation arguments.

Lemma 2.2. *Suppose $\phi : G(F)\backslash\mathbf{G} \to \mathbb{C}$ is slowly increasing and sufficiently smooth relative to* $\dim U_0$. *Then the alternating sum*

$$c\phi(g) := \sum_{P=MU\in\mathcal{F}(P_0)} (-1)^{a_M^G}\phi_P(g)$$

is rapidly decreasing. Moreover c extends to a projection on $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ whose restriction to $L_0^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ is the identity. Here $a_M^G := \dim\mathfrak{a}_M/\mathfrak{a}_G$.

The proof is a simple extension of the argument showing that the classical holomorphic cusp forms are rapidly decreasing. From this, we see that the kernel of the restriction $R_0(f)$ of $R(f)$ to $L_0^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ is $c_yK(x,y)$ (c_y means the operator c applied in y) which is rapidly decreasing. Hence $R_0(f)$ is of Hilbert-Schmidt class and R_0 decomposes *discretely*. Moreover each irreducible component has finite multiplicity:

$$L_0^2(G(F)\mathfrak{A}_G\backslash\mathbf{G}) = \bigoplus_{\pi\in\Pi(\mathbf{G})} \pi^{\oplus m_{\mathrm{cusp}}(\pi)}.$$

2.3 Decomposition by cuspidal data

A pair (M, ρ) of $M \in \mathcal{L}(P_0)$ and an irreducible component ρ of $L^2_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})$ is called a *cuspidal pair* for G. We write $L^2_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\rho$ for the ρ-isotypic subspace in $L^2_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})$ and $\mathcal{A}_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\rho$ for its intersection with $\mathcal{A}_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})$. This is the underlying admissible $(\mathfrak{g}_\infty, \mathbf{K}_\infty) \times \mathbf{G}_f$-module of the unitary representation $L^2_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\rho$. A $G(F)$-conjugacy class of cuspidal pairs is a *cuspidal datum* for G. We write $\mathfrak{X}(G)$ for the set of cuspidal data for G.

Fix a cuspidal pair $(M, \rho) \in \mathfrak{X} \in \mathfrak{X}(G)$ and a finite set of $\mathbf{K}$-types $\mathfrak{F}$. Write $\widehat{P}^{\mathfrak{F}}_{(M,\rho)}$ for the space of smooth functions $\widehat{\phi} : \mathbf{U}M(F)\mathfrak{A}_G\backslash\mathbf{G} \to \mathbb{C}$ such that

(1) $M(F)\mathfrak{A}_M\backslash\mathbf{M} \ni m \mapsto \widehat{\phi}(mg) \in \mathbb{C}$ belongs to $\mathcal{A}_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\rho$ for any $g \in \mathbf{G}$;

(2) $\mathfrak{A}_M/\mathfrak{A}_G \ni a \longmapsto \widehat{\phi}(ag) \in \mathbb{C}$ is compactly supported for any $g \in \mathbf{G}$;

(3) $\mathbf{K} \ni k \longmapsto \widehat{\phi}(gk) \in \mathbb{C}$ belongs to the linear span of the matrix coefficients of $\mathbf{K}$-types in $\mathfrak{F}$ for any $g \in \mathbf{G}$.

Noting that $\widehat{\phi} \in \widehat{P}^{\mathfrak{F}}_{(M,\rho)}$ is rapidly decreasing, we deduce the following.

Lemma 2.3. *(i) For $\widehat{\phi} \in \widehat{P}^{\mathfrak{F}}_{(M,\rho)}$,*

$$\theta_\phi(g) := \sum_{\gamma \in P(F)\backslash G(F)} \widehat{\phi}(\gamma g)$$

converges absolutely and belongs to $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$.
(ii) If we write $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{\mathfrak{X}}$ for the closure of the span of $\bigcup_{\mathfrak{F}} \bigcup_{(M,\rho)\in\mathfrak{X}} \{\theta_\phi \mid \phi \in \widehat{P}^{\mathfrak{F}}_{(M,\rho)}\}$, then we have the orthogonal decomposition

$$L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G}) = \bigoplus_{\mathfrak{X} \in \mathfrak{X}(G)} L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{\mathfrak{X}}.$$

2.4 Cuspidal Eisenstein series

Next we set $P^{\mathfrak{F}}_{(M,\rho)}$ for the space of functions $\phi : (\mathfrak{a}^G_M)^*_{\mathbb{C}} \times \mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G} \to \mathbb{C}$ satisfying

(1) $(\mathfrak{a}^G_M)^*_{\mathbb{C}} \ni \lambda \mapsto \phi(\lambda; g) \in \mathbb{C}$ is of Paley-Wiener type for any $g \in \mathbf{M}^1\mathbf{K}$;

(2) $M(F)\mathfrak{A}_M\backslash\mathbf{M} \ni m \mapsto \phi(\lambda; mk) \in \mathbb{C}$ belongs to $\mathcal{A}_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\rho$ for any $\lambda \in (\mathfrak{a}^G_M)^*_{\mathbb{C}}$, $k \in \mathbf{K}$;

(3) $\mathbf{K} \ni k \longmapsto \phi(\lambda; mk) \in \mathbb{C}$ belongs to the linear span of the matrix coefficients of $\mathbf{K}$-types in $\mathfrak{F}$ for any $\lambda \in (\mathfrak{a}^G_M)^*_{\mathbb{C}}$, $m \in \mathbf{M}^1$.

Obviously, any $\widehat{\phi} \in \widehat{P}^{\mathfrak{F}}_{(M,\rho)}$ is the Fourier transform

$$\widehat{\phi}(uamk) := \int_{i(\mathfrak{a}_M^G)^*} \phi(\lambda; mk) a^{\lambda}\, d\lambda \tag{2.2}$$

of some $\phi \in P^{\mathfrak{F}}_{(M,\rho)}$. On the other hand, associated to each $\phi \in P^{\mathfrak{F}}_{(M,\rho)}$ is a "Paley-Wiener section"

$$(\mathfrak{a}_M^G)^*_{\mathbb{C}} \ni \lambda \longmapsto [g \mapsto \phi_\lambda(g) := e^{\langle \lambda+\rho_P, H_P(g)\rangle}\phi(\lambda; g)] \in \mathcal{I}_P^G(\rho_\lambda)$$

of the bundle of induced representations $\mathcal{I}_P^G(\rho_\lambda) := \mathrm{ind}_{\mathbf{P}}^{\mathbf{G}}[(e^\lambda \otimes \rho) \otimes \mathbf{1}_{\mathbf{U}}] \to \lambda$. Here $\rho_P \in \mathfrak{a}_M^*$ denotes the half of the sum of positive roots of A_M in P.

For $\phi \in P^{\mathfrak{F}}_{(M,\rho)}$, the associated *cuspidal Eisenstein series* is defined by

$$E_P(x, \phi_\lambda) := \sum_{\delta \in P(F)\backslash G(F)} \phi_\lambda(\delta x). \tag{2.3}$$

$P = MU$, $P' = M'U' \in \mathcal{F}(P_0)$ are said to be *associated* if the set $W(M, M') := \{w \in W \,|\, w(M) = M'\}$ is not empty. Obviously the Weyl group W^M of M acts on this set by the right translation. A system of representatives of W^M-orbits in $W(M, M')$ is given by $W_{M,M'} := \{w \in W(M, M') \,|\, w(P_0^M) \subset P_0\}$. For $w \in W_{M,M'}$ we define the *intertwining operator* by the integral

$$[M(w, \rho_\lambda)\phi](x) := \int_{(\mathbf{U}'\cap w(\mathbf{U}))\backslash \mathbf{U}'} \phi_\lambda(w^{-1}ux)\, du \cdot e^{-\langle w(\lambda)+\rho_{P'}, H_{P'}(x)\rangle}.$$

Using the theory of resolvent, Langlands established the following properties [31], [33, Chapt. II, IV].

(1) Convergence. $E_P(x, \phi_\lambda)$ and $M(w, \rho_\lambda)\phi$ absolutely converge for $\mathrm{Re}(\lambda) >> 0$. At such λ, $E_P(x, \phi_\lambda)$ defines and automorphic form on $G(F)\mathfrak{A}_G\backslash \mathbf{G}$, and $\phi_\lambda \mapsto (M(w, \rho_\lambda)\phi)_{w(\lambda)}$ defines an intertwining operator $\mathcal{I}_P^G(\rho_\lambda) \to \mathcal{I}_{P'}^G(w(\rho_\lambda))$. Moreover the following holds.

- **(i) Equivariance.** $E_P(x, \mathcal{I}_P^G(\rho_\lambda, f)\phi_\lambda) = R(f)E_P(x, \phi_\lambda)$ for any $f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$.
- **(ii) Constant terms.** The constant term of $E_P(x, \phi_\lambda)$ along $Q = LV \in \mathcal{F}(P_0)$ is given by
$$E_P(x, \phi_\lambda)_Q = \sum_{w \in W_M(L)} E^L_{P^L_w}(x, (M(w, \rho_\lambda)\phi)_{w(\lambda)}).$$
Here $W_M(L) := \bigcup_{M' \in \mathcal{L}^L(P_0^L)} W_{M,M'}$ and P_w^L denotes the unique element of $\mathcal{F}^L(P_0^L)$ having $M_w := w(M)$ as its Levi component.
- **(iii) Functional equations.** $E_P(x, (M(w, \rho_\lambda)\phi)_{w\lambda}) = E_P(x, \phi_\lambda)$ for $w \in W_M(G)$. Also for $w \in W_{M,M'}$, $w' \in W_{M',M''}$, we have $M(w', w(\rho_\lambda))M(w, \rho_\lambda)\phi = M(w'w, \rho_\lambda)\phi$ at λ where both operators converge absolutely.
- **(iv) Fourier transform.** For λ_0 whose real part is sufficiently positive, we have
$$\theta_\phi(uamk) = \int_{\lambda_0 + i(\mathfrak{a}_M^G)^*} E_P(umk, \phi_\lambda) a^{\lambda+\rho_P}\, d\lambda.$$

(2) Analytic continuation. $E_P(x,\phi_\lambda)$ and $M(w,\rho_\lambda)\phi$ extend meromorphically to the whole $(\mathfrak{a}_M^G)^*_{\mathbb{C}}$. The properties (i) – (iv) of (1) still hold as equalities of meromorphic functions.

(3) Singularities. The set of poles of $E_P(x,\phi_\lambda)$ (hence those of $M(w,\rho_\lambda)\phi$ by (1-ii)) is a union of locally finite collection of affine hyperplanes whose vector parts are zeroes of some coroots.

2.5 Residual Eisenstein series and the spectral decomposition

Essentially by the Perseval formula, we deduced from (ii), (iv) the L^2-inner product formula:

$$\langle\theta_\phi,\theta_{\phi'}\rangle_G = \int_{\lambda_0+i(\mathfrak{a}_M^G)^*} \sum_{w\in W_{M,M'}} \langle M(w,\rho_\lambda)\phi,\phi'\rangle_M \, d\lambda. \tag{2.4}$$

Here $\langle\,,\,\rangle_G$ denotes the hermitian inner product on $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$. That is, the inner product between two θ_ϕ's is the integral over the Pontrjagin dual of $\mathfrak{A}_M^G$ of the Petersson inner product of $M(w,\rho_\lambda)\phi$ and ϕ'. Certain residue analysis transforms (2.4) into

$$\begin{aligned}\langle\theta_\phi,\theta_{\phi'}\rangle =_T &\int_{i(\mathfrak{a}_M^G)^*} \sum_{w\in W_{M,M'}} \langle M(w,\rho_\lambda)\phi,\phi'\rangle_M \, d\lambda \\ &+\sum_{\mathfrak{S}} \int_{o(\mathfrak{S})+i(\mathfrak{a}_{M_\mathfrak{S}}^G)^*} \sum_{w\in W_{M,M'}} \operatorname{Res}_\mathfrak{S} \langle M(w,\rho_\lambda)\phi,\phi'\rangle_M \, d\lambda.\end{aligned} \tag{2.5}$$

$=_T$ denotes a certain equivalence relation, the sum on the right runs over a finite set of intersections $\mathfrak{S}$ of singular hyperplanes of $E_P(x,\phi_\lambda)$ and $o(\mathfrak{S})$ is a certain "origin" of $\mathfrak{S}$. $M_\mathfrak{S}\in\mathcal{L}(M_0)$ is such that $i(\mathfrak{a}_{M_\mathfrak{S}}^G)^*$ equals the vector part of $\mathfrak{S}$. $\operatorname{Res}_\mathfrak{S}$ means the iterated residue along $\mathfrak{S}$. Noting that $\sum_{w\in W_{M,M'}}\operatorname{Res}_\mathfrak{S} M(w,\rho_\lambda)\phi$ belongs to the discrete spectrum of $L^2(M_\mathfrak{S}(F)\mathfrak{A}_{M_\mathfrak{S}}\backslash\mathbf{M}_\mathfrak{S})$ and, at the same time, equals to the constant term of certain residual Eisenstein series, we arrive at the following theorem [31, Chapt. 7], [33, Chapt. 6]

Theorem 2.4. *We call a pair (M,π) consisting of $M\in\mathcal{L}(P_0)$ and an irreducible subrepresentation π of $L^2(M(F)\mathfrak{A}_M\backslash\mathbf{M})$ a* discrete pair. *A $G(F)$-conjugacy class $[M,\pi]$ of discrete pairs is a* discrete datum. *Write $[P]$ for the associated class of $P\in\mathcal{F}(P_0)$.*
(1) For a discrete pair (M,π) and a finite set of $\mathbf{K}$-types $\mathfrak{F}$, we define the space $P_{(M,\pi)}^\mathfrak{F}$ in the same manner as $P_{(M,\rho)}^\mathfrak{F}$ with $\mathcal{A}_0(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\rho$ replaced by $\mathcal{A}_2(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\pi$, the intersection of the space of square-integrable automorphic forms $\mathcal{A}_2(M(F)\mathfrak{A}_M\backslash\mathbf{M})$ with the π-isotypic subspace $L^2(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\pi$. Then the residual Eisenstein series $E_P(x,\phi_\lambda)$ *defined by (2.3) with $\phi\in P_{(M,\pi)}^\mathfrak{F}$ satisfies the properties (1), (2) and (3) of 2.4 (But this time, the formula (1-ii) holds only for $Q\supset M$.).*
(2) Let $\widehat{L}_{[M,\pi]}$ for the Hilbert space of families of functions $F=\{F_{P'}\}_{P'\in[P]}$ such that

(i) $F_{P'}: i(\mathfrak{a}_{M'}^G)^*\to L^2(\mathbf{U}'M'(F)\mathfrak{A}_{M'}\backslash\mathbf{M}')_{\pi'}$ is a measurable function. Here $(M',\pi')\in[M,\pi]$.

(ii) $F_{P'}(w(\lambda)) = M(w,\pi_\lambda)F_P(\lambda)$, $w \in W_{M,M'}$.

(iii) The norm

$$\|F\|^2 := \sum_{(M',\pi')\in[M,\pi]} \frac{1}{|W_M(G)|} \int_{i(\mathfrak{a}_{M'}^G)^*} \|F_{P'}(\lambda')\|^2_{M'}\, d\lambda'$$

is finite.

Then there exists a $\mathbf{G}$*-equivariant unitary injection* $\widehat{L}_{[M,\pi]} \hookrightarrow L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$*, whose restriction to the automorphic spectrum valued Paley-Wiener sections is given by*

$$F \longmapsto \sum_{(M',\pi')\in[M,\pi]} \frac{1}{|W_M(G)|} \int_{i(\mathfrak{a}_{M'}^G)^*} E_{P'}(x, F_{P'}(\lambda'))\, d\lambda'.$$

If we write $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{[\pi]}$ *for the image of this map, we have the orthogonal decomposition*

$$L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G}) = \bigoplus_{[M,\pi]} L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{[\pi]}.$$

Note that the properties of residual Eisenstein series are deduced from their realization as residues of cuspidal Eisenstein series. In particular, various growth properties of cuspidal Eisenstein series are no longer assured for residual ones. This makes the construction of the trace formula much harder in the case when the F-rank is more than one.

2.6 Spectral kernel

Th. 2.4 allows us to deduce the spectral expression of the kernel $K(x,y)$ (2.1).

Choose a discrete datum $[M,\pi]$, a cuspidal datum $\mathfrak{X} \in \mathfrak{X}(G)$ and a finite set of $\mathbf{K}$-types $\mathfrak{F}$. Set

$$\mathcal{A}_2(M(F)\mathfrak{A}_M\backslash\mathbf{M})_{\pi,\mathfrak{X}} := \mathcal{A}_2(M(F)\mathfrak{A}_M\backslash\mathbf{M})_\pi \cap \bigoplus_{\mathfrak{X}^M\in\mathfrak{X}(M)\cap\mathfrak{X}} L^2(M(F)\mathfrak{A}_M\backslash\mathbf{M})_{\mathfrak{X}^M},$$

and write $\mathcal{A}_2(M(F)\mathfrak{A}_M\backslash\mathbf{M})^{\mathfrak{F}^M}_{\pi,\mathfrak{X}}$ for the subspace of functions which transform under $\mathfrak{F}^M$, the set of irreducible components of the restrictions to $\mathbf{K}^M := \mathbf{K}\cap\mathbf{M}$ of elements in $\mathfrak{F}$, under $\mathbf{K}^M$. This is a finite dimensional subspace of $L^2(M(F)\mathfrak{A}_M\backslash\mathbf{M})$, so that we can choose a ortho-normal basis $\mathfrak{B}^{\mathfrak{F}^M}_{\pi,\mathfrak{X}}$ of it satisfying $\mathfrak{B}^{\mathfrak{F}^M}_{\pi,\mathfrak{X}} \subset \mathfrak{B}^{\mathfrak{F}'^M}_{\pi,\mathfrak{X}}$ if $\mathfrak{F} \subset \mathfrak{F}'$.Similarly we have an orthonormal basis $\mathfrak{B}^{\mathfrak{F}}_{\pi,\mathfrak{X}}$ of the induced space $\mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})^{\mathfrak{F}}_{\pi,\mathfrak{X}} = \mathrm{Ind}^{\mathbf{K}}_{\mathbf{K}^M}\mathcal{A}_2(M(F)\mathfrak{A}_M\backslash\mathbf{M})^{\mathfrak{F}^M}_{\pi,\mathfrak{X}}$.
Associated to each $\varphi \in \mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})^{\mathfrak{F}}_{\pi,\mathfrak{X}}$ is the "constant section"

$$\varphi_\lambda(uamk) := a^{\lambda+\rho_P}\varphi(mk)$$

of the bundle $\mathcal{I}^G_P(\pi_\lambda) \to \lambda \in (\mathfrak{a}^G_M)^*_{\mathbb{C}}$.

Now the space $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})^{\mathfrak{F}}_{[\pi],\mathfrak{X}}$ for functions transforming according to $\mathfrak{F}$ under $\mathbf{K}$ in

$$L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{[\pi]} \cap L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{\mathfrak{X}}$$

is spanned by

$$\Phi_F(x) := \int_{i(\mathfrak{a}_M^G)^*} E(x, F(\lambda)_\lambda)\, d\lambda,$$

where $F : i(\mathfrak{a}_M^G)^* \longrightarrow \mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}}^{\mathfrak{F}}$ is a Paley-Wiener section. Using the expansion $F(\lambda) = \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}} \langle F(\lambda)_\lambda, \varphi_\lambda\rangle\varphi_\lambda$, we have

$$\begin{aligned}[R(f)\Phi_F](x) &= R(f)\int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}} \langle F(\lambda)_\lambda, \varphi_\lambda\rangle E_P(x,\varphi_\lambda)\, d\lambda \\ &= \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}} R(f)E_P(x,\varphi_\lambda)\langle F(\lambda)_\lambda, \varphi_\lambda\rangle\, d\lambda,\end{aligned}$$

for $f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)^{\mathfrak{F}}$. Now recall the $\mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$-equivariance

$$R(f)E_P(x,\varphi_\lambda) = E_P(x, \mathcal{I}_P^G(\pi_\lambda, f)\varphi_\lambda)$$

and the following formula obtained by the Fourier inversion formula applied to $\mathfrak{A}_M/\mathfrak{A}_G \times (\mathfrak{a}_M^G)^*$:

$$\langle F(\lambda)_\lambda, \varphi_\lambda\rangle = \langle \int_{i(\mathfrak{a}_M^G)^*} E_P(F(\mu)_\mu)\, d\mu, E_P(\varphi_\lambda)\rangle_G.$$

We see that $[R(f)\Phi_F](x)$ equals to

$$\begin{aligned}&\int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}} E_P(x, \mathcal{I}_P^G(\pi_\lambda, f)\varphi_\lambda)\langle \int_{i(\mathfrak{a}_M^G)^*} E_P(F(\mu)_\mu)\, d\mu, E_P(\varphi_\lambda)\rangle_G\, d\lambda \\ &\qquad = \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}} E_P(x, \mathcal{I}_P^G(\pi_\lambda, f)\varphi_\lambda)\overline{E_P(y,\varphi_\lambda)}\, d\lambda \Phi_F(y)\, dy.\end{aligned}$$

In other words, $R(f)$ $(f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)^{\mathfrak{F}})$ restricted to $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{[\pi],\mathfrak{X}}^{\mathfrak{F}}$ has the kernel

$$K_{[\pi],\mathfrak{X}}^{\mathfrak{F}}(x,y) := \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}} E_P(x, \mathcal{I}_P^G(\pi_\lambda, f)\varphi_\lambda)\overline{E_P(y,\varphi_\lambda)}\, d\lambda.$$

It was shown in [2] that

$$\begin{aligned}K_{\mathfrak{X}}(x,y) := &\sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \frac{1}{|W_M(G)|} \\ &\int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathfrak{B}_{\pi,\mathfrak{X}}} E_P(x, \mathcal{I}_P^G(\pi_\lambda, f)\varphi_\lambda)\overline{E_P(y,\varphi_\lambda)}\, d\lambda,\end{aligned} \tag{2.6}$$

defined as a certain limit, exists. Here $\mathfrak{B}_{\pi,\mathfrak{X}} := \bigcup_{\mathfrak{F}} \mathfrak{B}_{\pi,\mathfrak{X}}^{\mathfrak{F}}$ is empty if π does not appear in the discrete spectrum of $L^2(M(F)\mathfrak{A}_M\backslash\mathbf{M})$. Also the factor $1/|W_M(G)|$ is necessary because we take the sum over all the standard parabolics, not over their associated classes. We obtain the *spectral kernel*:

$$K(x,y) = \sum_{\mathfrak{X}\in\mathfrak{X}(G)} K_{\mathfrak{X}}(x,y). \tag{2.7}$$

3 Convergence

3.1 Truncated kernel

One can easily see that the operator $R(f)$ is no longer of trace class and

$$\int_{G(F)\mathfrak{A}_G\backslash \mathbf{G}} K(x,x)\,dx$$

diverges. The first step towards the trace formula is to truncate the kernel so that its integral over the diagonal converges. To see how this works, we need the following combinatorial functions.

We write Δ_0 and $\Delta_0^\vee$ for the set of simple roots of A_0 in P_0 and the set of corresponding simple coroots, respectively. For $P = MU \in \mathcal{F}(P_0)$, set $\Delta_P := \{\alpha|_{\mathfrak{A}_M} \mid \alpha \in \Delta_0 \setminus \Delta_0^M\}$ and $\Delta_P^\vee := \{(\alpha^\vee)_M \mid \alpha \in \Delta_0 \setminus \Delta_0^M\}$, where X_M denotes the $\mathfrak{a}_M$ component of $X = X^M \oplus X_M \in \mathfrak{a}_0$ under the W-invariant decomposition $\mathfrak{a}_0 = \mathfrak{a}_0^M \oplus \mathfrak{a}_M$. Using this, we define the positive chamber

$$\mathfrak{a}_P^+ := \{X \in \mathfrak{a}_0 \mid \alpha(X_M) > 0,\ \forall \alpha \in \Delta_P\},$$

whose characteristic function is denoted by τ_P. Next let $\widehat{\Delta}_P \subset (\mathfrak{a}_M^G)^*$ be the dual basis of the basis $\Delta_P^\vee \subset \mathfrak{a}_M^G$. Then the convex open cone

$$^+\mathfrak{a}_P := \{X \in \mathfrak{a}_0 \mid \varpi(X) = \varpi(X_M) > 0,\ \forall \varpi \in \widehat{\Delta}_P\}$$

is spanned by $\Delta_P^\vee$ (Strictly speaking, this is the direct sum of $\mathfrak{a}_G$ with an open cone.). Write $\widehat{\tau}_P$ for the characteristic function of this cone.

Let $\mathfrak{O}(G)$ be the set of semisimple classes in $G(F)$. For each $\mathfrak{o} \in \mathfrak{O}(G)$, we write $\bar{\mathfrak{o}}$ for the set of $\gamma \in G(F)$ whose semisimple part γ_s under the Jordan decomposition $\gamma = \gamma_s\gamma_u = \gamma_u\gamma_s$ belongs to $\mathfrak{o}$. We have

$$K(x,y) = \sum_{\mathfrak{o}\in\mathfrak{O}(G)} K_{\mathfrak{o}}(x,y), \quad K_{\mathfrak{o}}(x,y) := \sum_{\gamma\in\bar{\mathfrak{o}}} f(x^{-1}\gamma y).$$

Next, for $P = MU \in \mathcal{F}(P_0)$ write

$$K_P(x,y) = \sum_{\mathfrak{o}\in\mathfrak{O}(G)} K_{P,\mathfrak{o}}(x,y), \quad K_{P,\mathfrak{o}}(x,y) := \sum_{\gamma\in P(F)\cap\bar{\mathfrak{o}}} \int_{U(F)\backslash\mathbf{U}} f(x^{-1}\gamma u y)\,du$$

for the kernel of the right translation operator $R_P(f)$ on $L^2(\mathbf{U}M(F)\mathfrak{A}_G\backslash\mathbf{G})$ defined similarly as $R(f)$. Choosing $T \in \mathfrak{a}_0$, define the *truncated kernel*

$$k^T(x,f) := \sum_{\mathfrak{o}\in\mathfrak{O}(G)} k_{\mathfrak{o}}^T(x,f),$$

$$k_{\mathfrak{o}}^T(x,f) := \sum_{P=MU\in\mathcal{F}(P_0)} (-1)^{a_M^G} \sum_{\delta\in P(F)\backslash G(F)} K_{P,\mathfrak{o}}(\delta x,\delta x)\widehat{\tau}_P(H_0(\delta x) - T).$$

Example 3.1. *We present the explicit formulae in the case of $G = GL(2)$ [22]. The minimal parabolic subgroup will be denoted by $B = TU$. The $k_{\mathfrak{o}}^T(x, f)$ is given as follows.*
(i) If $\mathfrak{o}$ is elliptic, then

$$\sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma x). \tag{3.1}$$

(ii) If $\mathfrak{o} \ni \begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}$ is hyperbolic regular, then

$$\begin{aligned}\sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma x) &- \sum_{\delta\in B(F)\backslash G(F)} \int_{\mathbf{U}} f(x^{-1}\delta^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix} u\delta x)\widehat{\tau}_B(H_0(\delta x)-T)\,du \\ &- \sum_{\delta\in B(F)\backslash G(F)} \int_{\mathbf{U}} f(x^{-1}\delta^{-1}\begin{pmatrix}\beta & \\ & \alpha\end{pmatrix} u\delta x)\widehat{\tau}_B(H_0(\delta x)-T)\,du.\end{aligned} \tag{3.2}$$

(iii) If $\mathfrak{o} = \{\zeta\cdot 1\}$, $\zeta\in F^\times$, then

$$\sum_{\gamma\in\overline{\mathfrak{o}}} f(\zeta x^{-1}\gamma x) - \sum_{\delta\in B(F)\backslash G(F)} \int_{\mathbf{U}} f(\zeta x^{-1}\delta^{-1}u\delta x)\widehat{\tau}_B(H_0(\delta x)-T)\,du. \tag{3.3}$$

Note that this last term unifies the unipotent and identity terms.

Theorem 3.2 ([2] Th. 7.1). *If we choose $T\in\mathfrak{a}_0$ sufficiently positive with respect to* supp$f$, *then*

$$J^T(f) := \sum_{\mathfrak{o}\in\mathfrak{O}(G)} J_{\mathfrak{o}}^T(f),\quad J_{\mathfrak{o}}^T(f) := \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} k_{\mathfrak{o}}^T(x,f)\,dx$$

converges absolutely.

Let us sketch the proof. In the higher rank case, the following combinatorial argument is fundamental.

§ 2.1 combined with the Iwasawa decomposition yields $\mathbf{G} = P(F)\mathfrak{S}_P(T_0)$, where $\mathfrak{S}_P(T_0) = \omega_2\omega_1\mathfrak{A}_0^M(T_0)\mathfrak{A}_M\mathbf{K}$. If we write $F_Q^P(\bullet, T)$ for the characteristic function of

$$Q(F)\{x\in\mathfrak{S}_P(T_0)\,|\,\varpi(H_0(x)-T)\le 0,\ \forall\varpi\in\widehat{\Delta}_0^L\}$$

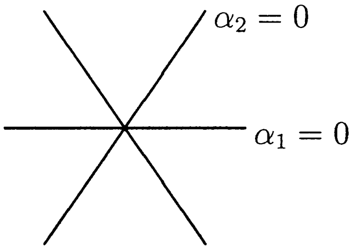
and $\mathfrak{J}_Q^P(\delta x, T) := F_Q^P(x,T)\tau_Q^P(H_0(x)-T)$, then we have [2, Lem. 6.4]:

$$\sum_{\substack{Q=LV\in\mathcal{F}(P_0)\\ Q\subset P}} \sum_{\delta\in Q(F)\backslash P(F)} \mathfrak{J}_Q^P(\delta x, T) = 1. \tag{3.4}$$

Next we put $\mathfrak{K}_Q^P(x,T) := F_Q^P(x,T)\sigma_Q^P(H_0(x)-T)$, where σ_Q^P is the characteristic function of

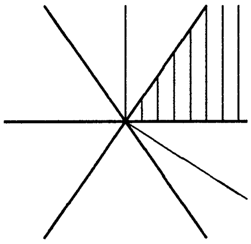
$$\left\{ X\in\mathfrak{a}_0 \;\middle|\; \begin{array}{ll} (1) & \alpha(X_L) > 0,\ \forall\alpha\in\Delta_{Q^M}, \\ (2) & \alpha(X_L)\le 0,\ \forall\alpha\in\Delta_Q\setminus\Delta_{Q^M}, \\ (3) & \varpi(X) > 0,\ \forall\varpi\in\widehat{\Delta}_Q. \end{array} \right\}$$

Example 3.3. *If $G = SL(3)$, the chambers in $\mathfrak{a}_0$ are illustrated as*

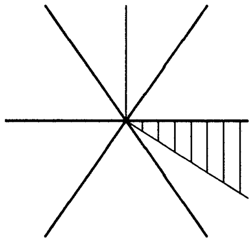


and σ_Q^P are the characteristic functions of the regions

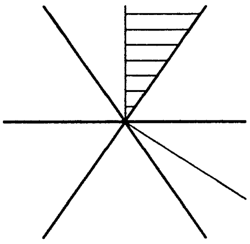
σ_0^G :



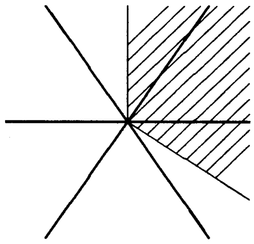
$\sigma_0^{P_1}$:



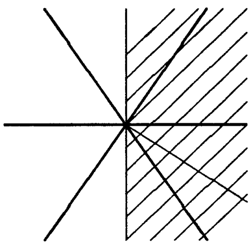
$\sigma_0^{P_2}$:



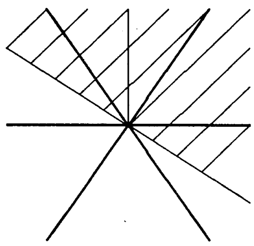
$\widehat{\tau}_0$:



$\widehat{\tau}_{P_1} = \sigma_{P_1}^G$:



$\widehat{\tau}_{P_2} = \sigma_{P_2}^G$:



where $P_i = M_i U_i \in \mathcal{F}(P_0)$ are such that $\Delta_{P_0^{M_i}} = \{\alpha_i\}$, $(i = 1,\, 2)$.

As these illustrate, we have $\sum_{R \supset P} \sigma_Q^R = \tau_{Q^M} \widehat{\tau}_P$ and hence

$$\sum_{\substack{R \in \mathcal{F}(P_0) \\ R \supset P}} \mathfrak{K}_Q^R(x, T) = \mathfrak{J}_Q^P(x, T). \tag{3.5}$$

Applying these combinatorial formulae to our integrand, we have

$$k_{\mathfrak{o}}^T(x,f) \overset{(3.4)}{=} \sum_{Q\subset P\in\mathcal{F}(P_0)} \sum_{\delta\in Q(F)\backslash G(F)} (-1)^{a_M^G} K_{P,\mathfrak{o}}(\delta x,\delta x)\mathfrak{J}_Q^P(\delta x,T)\widehat{\tau}_P(H_0(\delta x)-T)$$

$$\overset{(3.5)}{=} \sum_{Q\subset R\in\mathcal{F}(P_0)} \sum_{\delta\in Q(F)\backslash G(F)} \mathfrak{K}_Q^R(\delta x,T) \sum_{\substack{P\in\mathcal{F}(P_0)\\ Q\subset P\subset R}} (-1)^{a_M^G} K_{P,\mathfrak{o}}(\delta x,\delta x).$$

Thus it suffices to prove the convergence of

$$\int_{Q(F)\mathfrak{A}_G\backslash\mathbf{G}} |\mathfrak{K}_Q^R(x,T) \sum_{\substack{P\in\mathcal{F}(P_0)\\ Q\subset P\subset R}} (-1)^{a_M^G} K_P(x,x)|\,dx.$$

$\mathfrak{K}_Q^R(x,T)$ cut off the domain of integration to a product of a compact set and a transported positive cone in $\mathfrak{a}_L^{M_R}$. But on such cone the alternating sum in the integrand is rapidly decreasing by Lem. 2.2.

3.2 Truncation operator and the basic identity

Next we turn to the convergence of the spectral side [3]. Already it follows from Th. 3.2 that the integral of the total kernel

$$\int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} \sum_{\mathfrak{X}\in\mathfrak{X}(G)} k_{\mathfrak{X}}^T(x,f)\,dx$$

converges absolutely. Here we have written

$$k_{\mathfrak{X}}^T(x,f) := \sum_{P=MU\in\mathcal{F}(P_0)} (-1)^{a_M^G} \sum_{\delta\in P(F)\backslash G(F)} K_{P,\mathfrak{X}}(\delta x,\delta x)\widehat{\tau}_P(H_0(\delta x)-T),$$

where

$$K_{P,\mathfrak{X}}(x,y) = \sum_{\substack{Q=LV\in\mathcal{F}(P_0)\\ Q\subset P}} \sum_{\tau\in\Pi(\mathbf{L}^1)} \frac{1}{|W_L^M(M)|}$$
$$\int_{i(\mathfrak{a}_L^G)^*} \sum_{\varphi\in\mathfrak{B}_{\tau,\mathfrak{X}}} E_Q^P(x,\mathcal{I}_{Q^M}^M(\tau_\lambda,f)\varphi_\lambda)\overline{E_Q^P(y,\varphi_\lambda)}\,d\lambda$$

is the kernel of $R_P(f)$ restricted to $L^2(\mathbf{U}M(F)\mathfrak{A}_G\backslash\mathbf{G})_{\mathfrak{X}}$. Thus the problem is the commutativity of the integration and the summation.

For $T\in\mathfrak{a}_0$, define the *truncation operator* $\wedge^T$ on $L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$ by

$$(\wedge^T\phi)(x) := \sum_{P=MU\in\mathcal{F}(P_0)} (-1)^{a_M^G} \sum_{\delta\in P(F)\backslash G(F)} \phi_P(\delta x)\widehat{\tau}_P(H_0(\delta x)-T).$$

As in the proof of Th. 3.2, we can rewrite this as

$$(\wedge^T\phi)(x) = \sum_{Q\subset P_1\in\mathcal{F}(P_0)} \sum_{\delta\in Q(F)\backslash G(F)} \mathfrak{K}_Q^{P_1}(\delta x,T) \sum_{P;\,Q\subset P\subset P_1} (-1)^{a_M^G}\phi_P(\delta x),$$

and obtain the following properties:

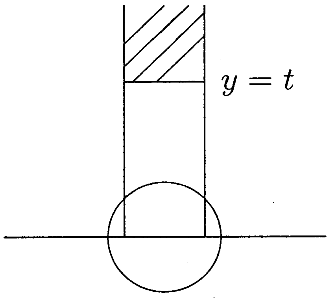
(1) $\wedge^T$ is an orthogonal projection.

(2) $\wedge^T\phi = \phi$ for $\phi \in L_0^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})$.

(3) If ϕ is slowly increasing and sufficiently smooth relative to $\dim U_0$, then $\wedge^T\phi$ is rapidly decreasing.

We have similar operators $\wedge^{T,P}$ on $L^2(\mathbf{U}M(F)\mathfrak{A}_G\backslash\mathbf{G})$.

Example 3.4. *Consider the simplest example, the $GL(2)$ case. Note that if T is sufficiently positive the sum on δ contains only one term. (Recall the classical situation where any element of $SL_2(\mathbb{Z})$ which preserves the region*



with sufficiently large t is upper triangular.) Then $\wedge^T$ is the operator which cuts off the constant term in the neighborhood $\{x \mid \widehat{\tau}_P(H_0(x) - T) > 0\}$ of cusps, the complement of a compact set in the Siegel domain. The properties stated above are rather obvious in this example.

A similar combinatorics as in the geometric side yield

$$k_{\mathfrak{X}}^T(x, f) = \sum_{Q \subset R \in \mathcal{F}(P_0)} \sum_{\delta \in Q(F)\backslash G(F)} \sigma_Q^R(H_0(\delta x) - T) \times \sum_{\substack{P \in \mathcal{F}(P_0) \\ Q \subset P \subset R}} (-1)^{a_M^G} \wedge_2^{T,Q} K_{P,\mathfrak{X}}(\delta x, \delta x).$$

Here $\wedge_2^T$ means the truncation operator is applied in the second variable. This combined with (3) above shows that the integration and the summation commute. Moreover the alternating sum on the right gives

$$\begin{aligned} J_{\mathfrak{X}}^T(f) &:= \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} k_{\mathfrak{X}}^T(x, f)\, dx \\ &= \sum_{Q \subset R \in \mathcal{F}(P_0)} \int_{Q(F)\mathfrak{A}_G\backslash\mathbf{G}} \sigma_Q^R(H_0(x) - T) \sum_{\substack{P \in \mathcal{F}(P_0) \\ Q \subset P \subset R}} (-1)^{a_M^G} \wedge^{T,Q} K_{P,\mathfrak{X}}(x, x)\, dx \qquad (3.6) \\ &= \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} \wedge^T K_{\mathfrak{X}}(x, x)\, dx. \end{aligned}$$

That is, the term associated to Q, R vanishes unless $Q = R = G$.

We now come to the coarse Arthur-Selberg trace formula

$$\sum_{\mathfrak{o}\in\mathfrak{O}(G)} J_{\mathfrak{o}}^T(f) = \sum_{\mathfrak{X}\in\mathfrak{X}(G)} J_{\mathfrak{X}}^T(f), \tag{3.7}$$

for sufficiently positive $T \in \mathfrak{a}_0$.

4 The fine $\mathfrak{O}$-expansion

Next we need to calculate each terms $J_{\mathfrak{o}}^T(f)$ and $J_{\mathfrak{X}}^T(f)$ in (3.7). Recall that, in the $GL(2)$ case [24], [22], certain part of the hyperbolic terms and Eisenstein terms cancel each other. The resulting equality has a meaning when $T \to \infty$. Since such an explicit cancellation is not available in the present case, taking the limit under $T \to \infty$ is not allowed. Instead we calculate the special value at "$T = 0$".

4.1 $J^T(f)$ as a polynomial

For $Q \subset P \in \mathcal{F}(P_0)$, consider the function

$$\Gamma_Q^P(X,Y) := \sum_{P_1;\, Q\subset P_1\subset P} (-1)^{a_{M_1}^M} \tau_{QM_1}(X)\widehat{\tau}_{P_1^M}(X-Y)$$

on $\mathfrak{a}_0 \times \mathfrak{a}_0$. Noting that the matrices $((-1)^{a_L}\tau_{QM})_{Q,P}$ and $((-1)^{a_L}\widehat{\tau}_{QM})_{Q,P}$ are inverse to each other, one sees that this function expresses the variation of $\widehat{\tau}_{QM}$:

$$\widehat{\tau}_{QM}(X-Y) = \sum_{P_1;\, Q\subset P_1\subset P} (-1)^{a_{M_1}^M} \widehat{\tau}_{QM_1}(X)\Gamma_{P_1}^P(X,Y). \tag{4.1}$$

Some calculation shows that

$$J_{\bullet}^{T+X}(f) = \sum_{P\in\mathcal{F}(P_0)} J_{\bullet M}^{M,T}(\overline{f}_P) \int_{\mathfrak{a}_M^G} \Gamma_P^G(H,X)\, dH,$$

where $\overline{f}_P$ is the descent

$$\overline{f}_P(m) := m^{\rho_P} \int_{\mathbf{K}} \int_{\mathbf{U}} f(k^{-1}muk)\, du\, dk$$

of f to M. Since the integral of $\Gamma_P^G(H,X)$ in H is a polynomial function in X, the same is true for $J_{\bullet}^T(f)$ [4, Prop. 2.3].

Recall that the distributions in (3.7) depend on the choice of the set of representatives $W \in \mathrm{Norm}(A_0, G)$, which can be chosen in $G(F) \cap \mathbf{K}$ if G is split. In general $W \subset \mathbf{KM}_0 \cap G(F)$ and, for $\alpha \in \Delta_0$, the representative w_{r_α} of the reflection r_α attached to α satisfies $H_0((w_{r_\alpha})^{-1}) = h_\alpha \alpha^\vee$ for some $h_\alpha \in \mathbb{R}$. We define the (analytic) origin $T_1 \in \mathfrak{a}_0$ by

$$T_1 := \sum_{\alpha\in\Delta_0} h_\alpha \varpi_\alpha^\vee.$$

Here $\{\varpi_\alpha^\vee\}_{\alpha\in\Delta_0}$ is the basis of $\mathfrak{a}_0^G$ dual to $\Delta_0 \in (\mathfrak{a}_0^G)^*$. We define $J_{\mathfrak{o}}(f) := J_{\mathfrak{o}}^{T_1}(f)$, $J_{\mathfrak{X}}(f) := J_{\mathfrak{X}}^{T_1}(f)$ [4, § 1, 2].

Example 4.1. *Let us compare this to the classical $GL(2)$ case. The distributions $J_{\mathfrak{o}}^T(f)$ are calculated as follows [22].*

(1) If $\mathfrak{o}$ is elliptic, we have

$$J_{\mathfrak{o}}^T(f) = \mathrm{meas}(G_\gamma(F)\mathfrak{A}_G\backslash\mathbf{G}_\gamma)\int_{\mathbf{G}_\gamma\backslash\mathbf{G}} f(x^{-1}\gamma x)\,dx \tag{4.2}$$

as in the anisotropic case. Note that we have $G^\gamma = G_\gamma$ in the special case of $GL(2)$.

(2) If $\mathfrak{o}$ is the hyperbolic regular class of $\left(\begin{smallmatrix}\alpha & \\ & \beta\end{smallmatrix}\right)$, (3.2) equals the sum over $\delta \in B(F)\backslash G(F)$ and $w \in W$ of

$$\frac{1}{2}\sum_{\nu\in U(F)} f\left(\mathrm{Ad}(w\nu\delta x)^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right) - \int_{\mathbf{U}} f\left(\mathrm{Ad}(wu\delta x)^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right) du\hat{\tau}_B(H_0(\delta x)-T).$$

One sees that $J_{\mathfrak{o}}^T(f)$ equals

$$\begin{aligned}
&\sum_{w\in W}\int_{B(F)\mathfrak{A}_G\backslash\mathbf{G}} \frac{1}{2}\sum_{\nu\in U(F)} f\left(\mathrm{Ad}(w\nu x)^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right)\\
&\qquad\qquad - \int_{\mathbf{U}} f\left(\mathrm{Ad}(wux)^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right) du\hat{\tau}_B(H_0(x)-T)\,dx\\
&=\sum_{w\in W}\int_{\mathbf{U}T(F)\mathfrak{A}_G\backslash\mathbf{G}}\int_{\mathbf{U}} f\left(\mathrm{Ad}(wux)^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right)\left(\frac{1}{2}-\hat{\tau}_B(H_0(x)-T)\right) du\,dx\\
&=\sum_{w\in W}\int_{T(F)\mathfrak{A}_G\backslash\mathbf{G}} f\left(w^{-1}x^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix} xw\right)\left(\frac{1}{2}-\hat{\tau}_B(H_0(x)-T)\right) dx\\
&=\int_{T(F)\mathfrak{A}_G\backslash\mathbf{G}} f\left(x^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix} x\right)\left(1-\hat{\tau}_B(H_0(x)-T)-\hat{\tau}_B(H_0(wx)-T)\right) dx
\end{aligned}$$

using the Iwasawa decomposition,

$$\begin{aligned}
&=\mathrm{meas}(T(F)\mathfrak{A}_G\backslash\mathbf{T}^1)\int_{\mathbf{K}}\int_{\mathbf{U}} f\left(k^{-1}u^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix} uk\right)\\
&\qquad\qquad\int_{\mathfrak{A}_T^G} 1-\hat{\tau}_B(H_0(a)-T)-\hat{\tau}_B(H_0(wau)-T)\,da\,du\,dk.
\end{aligned}$$

Noting $H_0(wau) = w(H_0(a)) + H_0(wu)$, the inner integral becomes

$$\int_{\alpha(H_0(wu)-T)\alpha^\vee/2}^{\alpha(T)\alpha^\vee/2} dH = \frac{\alpha(2T-H_0(wu))}{\sqrt{2}}.$$

Here α denotes the unique positive root of T in G. We obtain

$$J_{\mathfrak{o}}^T(f) = -\mathrm{meas}(T(F)\mathfrak{A}_G\backslash\mathbf{T}^1)\int_{\mathbf{K}}\int_{\mathbf{U}} f\left(k^{-1}u^{-1}\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix} uk\right)\frac{\alpha(H_0(wuk))}{\sqrt{2}}\,du\,dk \tag{4.3}$$

$$+\sqrt{2}\alpha(T)\mathrm{meas}(T(F)\mathfrak{A}_G\backslash\mathbf{T}^1)\overline{f}_P\left(\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right). \tag{4.4}$$

(3) The case of central $\mathfrak{o}$ *is easily reduced to the case* $\mathfrak{o} = \{1\}$. *If* $\mathfrak{o} = \{1\}$, $\overline{\mathfrak{o}} = \coprod_{\delta\in B(F)\backslash G(F)} \mathrm{Ad}(\delta)^{-1}U(F)$ *allows us to write* (3.3) *as*

$$f(1) + \sum_{\delta\in B(F)\backslash G(F)} \Big(\sum_{\nu\in U(F)\backslash\{1\}} f(x^{-1}\delta^{-1}\nu\delta x) - \int_{\mathbf{U}} f(x^{-1}\delta^{-1}u\delta x)\, du\widehat{\tau}_B(H_0(\delta x) - T)\Big).$$

Thus, by the Iwasawa decomposition, $J^T_{\{1\}}(f)$ *equals the sum of*

$$\mathrm{meas}(G(F)\mathfrak{A}_G\backslash\mathbf{G})f(1) \tag{4.5}$$

and

$$\int_{T(F)\mathfrak{A}_G\backslash\mathbf{T}} \left(\sum_{\nu\in U(F)\backslash\{1\}} \overline{f}(t^{-1}\nu t) - \int_{\mathbf{U}} \overline{f}(t^{-1}ut)\, du\widehat{\tau}_B(H_0(t) - T) \right) t^{-2\rho_B}\, dt, \tag{4.6}$$

where

$$\overline{f}(x) := \int_{\mathbf{K}} f(k^{-1}xk)\, dk.$$

Fix a non-trivial character ψ *of* $F\backslash\mathbb{A}$. *Write* $\mathfrak{u}$ *for the Lie algebra of* U *and* $\mathfrak{u}^\vee$ *be its dual. Then the Poisson summation formula for the Fourier transformation*

$$(\widehat{\mathrm{Ad}(t)\overline{f}\circ\exp})(X^\vee) := \int_{\mathfrak{u}(\mathbb{A})} (\mathrm{Ad}(t)\overline{f})(\exp X)\psi(\langle X^\vee, X\rangle)\, dX = t^{2\rho_B}(\widehat{\overline{f}\circ\exp})(\alpha(t)X^\vee)$$

implies that (4.6) *equals*

$$\begin{aligned}
&\sqrt{2}\cdot\mathrm{meas}(F^\times\backslash\mathbb{A}^1)\int_{F^\times\backslash\mathbb{A}^\times} \sum_{\xi\in\mathfrak{u}(F)\backslash\{0\}} (\overline{f}\circ\exp)(x^{-1}\xi)|x|_{\mathbb{A}}^{-1} - (\widehat{\overline{f}\circ\exp})(0)\tau_{>\alpha(T)}(\log|x|_{\mathbb{A}})\, dx^\times \\
&=\sqrt{2}\cdot\mathrm{meas}(F^\times\backslash\mathbb{A}^1)\Big(\int_{F^\times\backslash\mathbb{A}_{<1}} \sum_{\xi\in\mathfrak{u}(F)\backslash\{0\}} (\overline{f}\circ\exp)(x^{-1}\xi)|x|_{\mathbb{A}}^{-1}\, dx^\times \\
&\quad+\int_{F^\times\backslash\mathbb{A}_{\geq 1}} \sum_{\xi^\vee\in\mathfrak{u}^\vee(F)} (\widehat{\overline{f}\circ\exp})(x\xi^\vee) - \overline{f}(1)|x|_{\mathbb{A}}^{-1} - (\widehat{\overline{f}\circ\exp})(0)\tau_{>\alpha(T)}(\log|x|_{A})\, dx^\times\Big) \\
&=\sqrt{2}\mathrm{meas}(F^\times\backslash\mathbb{A}^1)\Big(\int_{F^\times\backslash\mathbb{A}_{>1}} \sum_{\xi\in\mathfrak{u}(F)\backslash\{0\}} (\overline{f}\circ\exp)(x\xi)|x|_{\mathbb{A}}\, dx^\times \\
&\qquad+\int_{F^\times\backslash\mathbb{A}_{\geq 1}} \sum_{\xi^\vee\in\mathfrak{u}^\vee(F)\backslash\{0\}} (\widehat{\overline{f}\circ\exp})(x\xi^\vee) - \overline{f}(1)|x|_{\mathbb{A}}^{-1}\, dx^\times \\
&\qquad+(\widehat{\overline{f}\circ\exp})(0)\int_{F^\times\backslash\mathbb{A}_{\geq 1}} 1 - \tau_{>\alpha(T)}(\log|x|_{\mathbb{A}})\, dx^\times\Big).
\end{aligned}$$

Here $\tau_{>A}$ *is the characteristic function of* $\mathbb{R}_{>A}$. *Consider the integral*

$$Z(\overline{f}, s) := \int_{\mathbb{A}^\times} (\overline{f}\circ\exp)(x)|x|_{\mathbb{A}}^s\, dx^\times.$$

This converges absolutely for $\mathrm{Re}(s) > 1$ *and extends to a meromorphic function in* s. *It turns out that the sum of the first two integrals in the above equals the regular part* $\mathrm{reg}_{s=1} Z(\overline{f}, s)$ *of this "zeta function" at* $s = 1$, *while the last equals* $\mathrm{meas}(F^\times\backslash\mathbb{A}^1)\alpha(T)$. *Noting that* $(\widehat{\overline{f}\circ\exp})(0) = \overline{f}_P(1)$, *we conclude that*

$$
\begin{aligned}
J^T_{\{1\}}(f) &= \mathrm{meas}(G(F)\mathfrak{A}_G\backslash\mathbf{G})f(1) + \mathrm{meas}(Z(F)\mathfrak{A}_G\backslash\mathbf{Z})\mathrm{reg}_{s=1} Z(\overline{f}, s) \qquad (4.7)\\
&\quad + \sqrt{2}\alpha(T)\mathrm{meas}(T(F)\mathfrak{A}_G\backslash\mathbf{T}^1)\overline{f}_P(1). \qquad (4.8)
\end{aligned}
$$

We choose $T_1 = 0$. *Then taking the special value at* T_1 *is equivalent to throw away the sum of* (4.4) *and* (4.8)*:*

$$
\sum_{\alpha,\beta\in F^\times} \sqrt{2}\alpha(T)\mathrm{meas}(T(F)\mathfrak{A}_G\backslash\mathbf{T}^1)\overline{f}_P\left(\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}\right).
$$

This is precisely the term which cancels the analogous term in the spectral side [22, p. 289]. Thus we can say that the present construction is a generalization of the $GL(2)$ *case. Note also that this term can be viewed as the geometric side of the "trace formula" for the Levi subgroup* T *applied to* $\overline{f}_P$.

4.2 Reduction by the Jordan decomposition

Calculate the geometric terms. Our goal is to express $J_{\mathfrak{o}}(f)$ as a sum of terms, each of which is a product of a global constant and an Euler product of local distributions.

First we replace $k^T_{\mathfrak{o}}(x, f)$ with

$$
j^T_{\mathfrak{o}}(x, f) := \sum_{P\in\mathcal{F}(P_0)} (-1)^{a^G_M} \sum_{\delta\in P(F)\backslash G(F)} j_{P,\mathfrak{o}}(\delta x)\widehat{\tau}_P(H_0(\delta x) - T),
$$

$$
j_{P,\mathfrak{o}}(x) := \sum_{\gamma\in M(F)\cap\overline{\mathfrak{o}}} \sum_{\nu\in U^{\gamma_s}(F)\backslash U(F)} \int_{\mathbf{U}^{\gamma_s}} \phi(x^{-1}\nu^{-1}\gamma u\nu x)\, du.
$$

We have

$$
K_{P,\mathfrak{o}}(x, x) = \int_{U(F)\backslash\mathbf{U}} j_{P,\mathfrak{o}}(ux)\, du
$$

and hence

$$
J^T_{\mathfrak{o}}(f) = \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} j^T_{\mathfrak{o}}(x, f)\, dx.
$$

Take a representative $\sigma \in \mathfrak{o}$ which is *elliptic* in some $P_1 = M_1U_1 \in \mathcal{F}(P_0)$, that is, $\sigma \in M_1(F)$ but it is not contained in any proper parabolic subgroup of M_1. Then $P_{1,\sigma}$ is a minimal parabolic subgroup of G_σ. Each $\gamma \in \overline{\mathfrak{o}}$ is conjugate to some element of $\sigma\mathcal{U}_{G_\sigma}(F)$, where $\mathcal{U}_{G_\sigma}$ is the unipotent variety of G_σ, the connected centralizer of σ. Thus writing $\iota^G(\sigma) := [G^\sigma(F) : G_\sigma(F)]$, we have

$$
\begin{aligned}
j_{Q,\mathfrak{o}}(x) =&\iota^G(\sigma)^{-1} \sum_{w\in W_{M_1}(L,G_\sigma)}\\
&\sum_{\pi\in({}^wG_\sigma\cap Q)(F)\backslash Q(F)} \sum_{\nu\in w^{-1}(L)(F)\cap\mathcal{U}_{G_\sigma}(F)} \int_{w^{-1}(\mathbf{V})\cap\mathbf{G}_\sigma} f(x^{-1}\pi^{-1w}(\sigma\nu v)\pi x)\, dv.
\end{aligned}
$$

Here

$$W_{M_1}(L,G_\sigma) := \left\{ w \in W_{M_1}(L) \;\middle|\; \begin{array}{ll} (i) & w^{-1}(\alpha) > 0,\ \forall\alpha \in \Delta_{{}^{w}P_1^L}, \\ (ii) & w(\beta) > 0,\ \forall\beta \in \Sigma_{P_{1,\sigma}} \end{array} \right\},$$

Σ_P being the set of positive roots of P. If we write R for the standard parabolic subgroup ${}^{w^{-1}}Q \cap G_\sigma$ of G_σ and $\mathcal{F}(M_1)_R := \{P \in \mathcal{F}(M_1) \mid P_\sigma = R\}$, the above gives

$$\begin{aligned} J_{\mathfrak{o}}^T(f) = &\iota^G(\sigma)^{-1} \int_{\mathbf{G}_\sigma\backslash\mathbf{G}} \int_{G_\sigma(F)\mathfrak{A}_G\backslash\mathbf{G}_\sigma} \sum_{R \in \mathcal{F}(P_{1,\sigma})} \sum_{\delta \in R(F)\backslash G_\sigma(F)} \\ &\sum_{\nu \in \mathcal{U}_{M_R}(F)} \int_{\mathbf{U}_R} f(y^{-1}x^{-1}\delta^{-1}\sigma\nu v\delta xy)\, dv \\ &\sum_{P \in \mathcal{F}(M_1)_R} (-1)^{a_M^G} \widehat{\tau}_P(H_0(\delta xy) - w_P^{-1}(T - T_1) - T_1)\, dx\, dy. \end{aligned} \tag{4.9}$$

$w_P \in W$ is such that ${}^{w_P}P \in \mathcal{F}(P_0)$. $R = M_R U_R$ denotes the standard Levi decomposition of R. We shall write this in terms of the unipotent terms $J_{\{1\}}^{M_S,T}$ of the trace formula for the Levi subgroups of G_σ.

As in the case of G, we have the origin $T_{1,\sigma} \in \mathfrak{a}_{M_1}$ for G_σ. We write T_σ for the image of $T - T_1 + T_{1,\sigma}$ in $\mathfrak{a}_{M_1}$ and set

$$Y_P^T(x,y) := -H_P(k_{P_\sigma}(x)y) + w_P^{-1}(T - T_1) - T_\sigma + T_1, \quad P \in \mathcal{F}(M_1)_R.$$

Then the family $\mathcal{Y}_P^T(x,y) := \{Y_P^T(x,y) \mid P \in \mathcal{F}(M_1)_R\}$ is *compatible* in the sense of [9, § 4]. Generalizing (4.1), we see that the last line in (4.9) can be written as

$$\sum_{\substack{S \in \mathcal{F}^{G_\sigma}(M_{1,\sigma}) \\ S \supset R}} (-1)^{a_{M_R}^{M_S}} \widehat{\tau}_R^{M_S}(H_R(\delta x) - T_\sigma)\Gamma_S^G(H_S(\delta x) - T_\sigma, \mathcal{Y}_S^T(\delta x, y)),$$

where

$$\Gamma_R^G(X, \mathcal{Y}_R) := \sum_{\substack{S \in \mathcal{F}^{G_\sigma}(M_{1,\sigma}) \\ S \supset R}} \tau_R^{M_S}(X) \left(\sum_{Q \in \mathcal{F}(M_1)_S} (-1)^{a_L^G} \widehat{\tau}_Q(X - Y_Q) \right).$$

This combined with the Iwasawa decomposition yields

$$J_{\mathfrak{o}}^T(f) = \iota^G(\sigma)^{-1} \int_{\mathbf{G}_\sigma\backslash\mathbf{G}} \sum_{S \in \mathcal{F}^{G_\sigma}(P_{1,\sigma})} \Big(\int_{\mathbf{K}_\sigma} \int_{\mathfrak{A}_S^G} J_{\{1\}}^{M_S,T_\sigma}(\Phi_{S,a,k,y}^T)\, da\, dk \Big)\, dy, \tag{4.10}$$

with

$$\Phi_{S,a,k,y}^T(m) := m^{\rho_S} \int_{\mathbf{U}_S} f(y^{-1}\sigma k^{-1} muky)\, du \Gamma_S^G(H_S(a) - T_\sigma, \mathcal{Y}_S^T(k,y)).$$

Finally if we specialize this to $T = T_1$, $\Gamma_S^G(X, \mathcal{Y}_S^{T_1}(k,y))$ reduces to $\Gamma_Q^G(X, -H_Q(ky) + T_1 - T_{1,\sigma})$ for any $Q \in \mathcal{F}(M_1)_S$. Applying this to the integral over a in (4.10), we obtain

$$\begin{aligned} J_{\mathfrak{o}}(f) &= \iota^G(\sigma)^{-1} \int_{\mathbf{G}_\sigma\backslash\mathbf{G}} \sum_{S \in \mathcal{F}^{G_\sigma}(P_{1,\sigma})} J_{\{1\}}^{M_S}(\Phi_{S,y,T_1 - T_{1,\sigma}})\, dy \\ &= \iota^G(\sigma)^{-1} \int_{\mathbf{G}_\sigma\backslash\mathbf{G}} \Big(\sum_{R \in \mathcal{F}^{G_\sigma}(M_{1,\sigma})} \frac{|W^{M_R}|}{|W^{G_\sigma}|} J_{\{1\}}^{M_R}(\Phi_{R,y,T_1 - T_{1,\sigma}}) \Big)\, dy, \end{aligned} \tag{4.11}$$

where

$$\Phi_{S,y,T}(m) := m^{\rho_S} \int_{\mathbf{K}_\sigma} \int_{\mathbf{U}_S} f(y^{-1}\sigma k^{-1} muky) v'_S(ky,T)\, du\, dk,$$
$$v_{S'}(ky,T) := \sum_{Q \in \mathcal{F}(M_1)_S} \int_{\mathfrak{a}_L^G} \Gamma_Q^G(X, -H_Q(ky)+T)\, dX.$$

4.3 Weighted orbital integrals

Our goal here is to express (4.11) as a linear combination of weighted orbital integrals. We must first recall the notion of (G,M)-family.

For $Q \subset P \in \mathcal{F}(M_0)$, we define two denominator functions

$$\theta_Q^P(\lambda) := \frac{1}{\mathrm{meas}(\mathfrak{a}_L^M/\mathbb{Z}[\Delta_{QM}^\vee])} \prod_{\alpha \in \Delta_{QM}} \alpha^\vee(\lambda),$$
$$\widehat{\theta}_Q^P(\lambda) := \frac{1}{\mathrm{meas}(\mathfrak{a}_L^M/\mathbb{Z}[\widehat{\Delta}_{QM}^\vee])} \prod_{\varpi^\vee \in \widehat{\Delta}_{QM}^\vee} \varpi^\vee(\lambda).$$

Here $\widehat{\Delta}_P^\vee$ denotes the basis of $\mathfrak{a}_M^G$ dual to $\Delta_P \subset (\mathfrak{a}_M^G)^*$. These functions first appeared in the Fourier transform of $\Gamma_Q^G(X,Y)$ in X [4, Lem. 2.2]:

$$\int_{\mathfrak{a}_L^G} \Gamma_Q^G(X,Y) e^{\lambda(X)}\, dX = \sum_{P \supset Q} (-1)^{a_L^M} \frac{e^{\langle \lambda_M, Y\rangle}}{\widehat{\theta}_Q^P(\lambda)\theta_P^G(\lambda)},$$

but its real nature is to produce certain "difference"

$$c'_Q(\lambda) := \sum_{P_1 \supset Q} (-1)^{a_L^{M_1}} \frac{c_{P_1}(\lambda)}{\widehat{\theta}_Q^{P_1}(\lambda)\theta_{P_1}^G(\lambda)} \tag{4.12}$$

of a smooth function $c_P(\lambda)$ on $i\mathfrak{a}_M^*$, $Q \supset P$. Here $c_{P_1}(\lambda) := c_P(\lambda_{M_1})$, λ_{M_1} being the $i\mathfrak{a}_{M_1}^*$ component of λ under $i\mathfrak{a}_M^* = i(\mathfrak{a}_M^{M_1})^* \oplus i\mathfrak{a}_{M_1}^*$.

Example 4.2. *In the case of $GL(2)$, we have*

$$c'_B(\lambda) = \frac{\sqrt{2}}{\alpha^\vee(\lambda)}(c_B(\lambda) - c'_G(\lambda)) = \frac{\sqrt{2}}{\alpha^\vee(\lambda)}(c_B(\lambda) - c_B(\lambda_G)).$$

Since $\sqrt{2}$ is the length of $\alpha^\vee$, this is literally the difference of $c_B(\lambda)$ around λ_G.

A family $\{c_P(\lambda)\}_{P \in \mathcal{P}(M)}$ of smooth functions on $i\mathfrak{a}_M^*$ is a *(G,M)-family* if $c_P(\lambda) = c_{P'}(\lambda)$ for any P, $P' \in \mathcal{P}(M)$ whose associated chambers share a wall and λ in that wall. For a (G,M)-family $\{c_P(\lambda)\}_{P \in \mathcal{P}(M)}$, the function (4.12) is well-defined because it is independent of the choice of $P \in \mathcal{P}^{P_1}(M)$.

Lemma 4.3 ([4] Lem. 6.1, 6.2, 6.3). *Let $\{c_P(\lambda)\}_{P\in\mathcal{P}(M)}$ be a (G,M)-family.*
(1) $c'_Q(\lambda)$ extends to a smooth function on $i\mathfrak{a}_L^*$.
(2) The function

$$c_M^Q(\lambda) := \sum_{\substack{P\in\mathcal{P}(M)\\ P\subset Q}} \frac{c_P(\lambda)}{\theta_P^Q(\lambda)}$$

extends to a smooth function on $i\mathfrak{a}_M^*$.
(3) If $\{d_P(\lambda)\}_{\mathcal{P}(M)}$ is another (G,M)-family, $(cd)_P(\lambda) := c_P(\lambda)d_P(\lambda)$ form a (G,M)-family and we have

$$(cd)_M(\lambda) = \sum_{Q\in\mathcal{F}(M)} c_M^Q(\lambda)d'_Q(\lambda).$$

An example of (G,L)-family is the family $\{v_Q(\lambda,x) := e^{-\lambda(H_Q(x))}\}_{Q\in\mathcal{P}(L)}$ for $x\in\mathbf{G}$. If $P=MU\in\mathcal{F}(L)$, then we have the bijection

$$\mathcal{P}^P(L)\ni Q\longmapsto Q^M := Q\cap M\in\mathcal{P}^M(L).$$

Thus, from a (G,L)-family $\{c_Q(\lambda)\}_{Q\in\mathcal{P}(L)}$, one deduces a (M,L)-family $\{c_{Q^M}(\lambda) := c_Q(\lambda)\}_{Q^M\in\mathcal{P}^M(L)}$. Applying this to $\{v_Q(\lambda,x)\}_{Q\in\mathcal{P}(L)}$, we have the smooth function

$$v_L^P(\lambda,x) := \sum_{Q^M\in\mathcal{P}^M(L)} \frac{v_Q(\lambda,x)}{\theta_Q^P(\lambda)}$$

by Lem. 4.3 (2). This *depends on $P\in\mathcal{P}(M)$* (as the notation suggests) because the function H_Q depends not only Q^M but also Q. Set $v_L^P(x) := \lim_{\lambda\to 0} v_L^P(\lambda,x)$.

Let S be a finite set of places of F. We now define the *weighted orbital integral* $J_M(\gamma,f)$ for $M\in\mathcal{L}(M_0)$ and $\gamma\in\mathbf{M}_S$. Write $\gamma_v=\gamma_{v,s}\gamma_{v,u}$ for the Jordan decomposition of γ_v. If $G_{\gamma_{v,s}}\subset M_{\gamma_{v,s}}$ at any $v\in S$, we define

$$\begin{aligned} J_M(\gamma,f) &= |D^G(\gamma)|_{\mathbb{A}_S}^{1/2}\int_{\mathbf{M}_S\backslash\mathbf{G}_S}\int_{\mathfrak{o}^{\mathbf{M}_S}(\gamma)} f(x^{-1}\mu x)v_M(x)\,d\mu\,dx \\ &= |D^G(\gamma)|_{\mathbb{A}_S}^{1/2}\int_{\mathbf{G}_{S,\gamma}\backslash\mathbf{G}_S} f(x^{-1}\gamma x)v_M(x)\,dx. \end{aligned}$$

Here $D^G(\gamma)=(D^G(\gamma_v))_{v\in S}$ is given by $D^G(\gamma) := \det(1-\mathrm{Ad}(\gamma_{v,s})|\mathfrak{g}(F_v)/\mathfrak{g}_{\gamma_{v,s}}(F_v))$, $\mathfrak{g}_{\gamma_{v,s}}$ being the fixed part of $\mathrm{Ad}(\gamma_{v,s})$ in $\mathfrak{g}_v$, the Lie algebra of $G\otimes_F F_v$. Also $\mathfrak{o}^{\mathbf{M}_S}(\gamma)$ is the $\mathbf{M}_S$-orbit of γ. The convergence of $J_M(\gamma,f)$ is assured by [10, Lem. 2.1] and the Deligne-Rao theorem [36]. In the general case, we define

$$J_M(\gamma,f) := \lim_{a\to 1}\sum_{L\in\mathcal{L}(M)} r_M^L(\gamma,a)J_L(a\gamma,f). \tag{4.13}$$

Here $r_M^L(\gamma,a)=r_M^Q(\gamma,a)$ ($Q\in\mathcal{P}(L)$) is constructed from the (G,M)-family

$$r_P(\lambda,\gamma,a) = \prod_{v\in S}\prod_{\beta\in\Sigma^r_{P_{\gamma_{v,s}}}} |a^\beta - a^{-\beta}|_v^{\rho(\beta,\gamma_{v,u})\lambda(\beta^\vee)/2},\quad P\in\mathcal{P}(M)$$

by the process explained above. The non-negative integers $\rho(\beta, u)$ are defined in [10, § 3], and the existence of the limit in (4.13) is assured by [10, Th.5.2]. In this case the resulting function is independent of $Q \in \mathcal{P}(L)$ [10, Lem. 5.1]. Note that, if $M = G$ this reduces to the ordinary orbital integral

$$J_G(\gamma, f) = |D^G(\gamma)|_{\mathbb{A}_S}^{1/2} \int_{\mathbf{G}_{S,\gamma}\backslash \mathbf{G}_S} f(x^{-1}\gamma x)\, dx.$$

We need the following descent formula for weighted orbital integrals.

Lemma 4.4 ([10] Cor. 8.7). *Suppose that $\gamma \in \mathbf{G}_S$ satisfies*

(i) $(\gamma_{v,s})_{v\in S} \in G(F)$;

(ii) $\mathfrak{a}_{M_{\gamma_s}} = \mathfrak{a}_M$ *(γ_s is elliptic in $M(F)$).*

Then we have

$$J_M(\gamma, f) = |D^G(\gamma_s)|_{\mathbb{A}_S}^{1/2} \int_{\mathbf{G}_{\gamma_s,S}\backslash \mathbf{G}_S} \left(\sum_{R\in\mathcal{F}^{G_{\gamma_s}}(M_{\gamma_s})} J_{M_{\gamma_s}}^{M_R}(\gamma_u, \Phi_{R,y,T}) \right) dy,$$

where

$$\Phi_{R,y,T}(m) := m^{\rho_R} \int_{\mathbf{K}_{\gamma_s,S}} \int_{\mathbf{U}_{R,S}} f(y^{-1}\gamma_s k^{-1} muky) v_R'(ky, T)\, du\, dk$$

is the local analogue of $\Phi_{R,y,T}$ in (4.11). In particular the right hand side does not depend on T.

We now go back to the calculation of (4.11). Take a finite set S of places of F sufficiently large with respect to G and f so that

$$J_{\mathfrak{o}}(f) = \iota^G(\sigma)^{-1} \int_{\mathbf{G}_{\sigma,S}\backslash \mathbf{G}_S} \left(\sum_{R\in\mathcal{F}^{G_\sigma}(M_{1,\sigma})} \frac{|W^{M_R}|}{|W^{G_\sigma}|} J_{\{1\}}^{M_R}(\Phi_{R,y,T_1 - T_{1,\sigma}}) \right) dy \tag{4.14}$$

We have following formulae on the variation of $J_{\{1\}}(f)$ and $J_M(\gamma, f)$ under conjugations:

$$J_{\{1\}}(\mathrm{Ad}(g^{-1})f) = \sum_{Q\in\mathcal{F}(M_0)} \frac{|W^L|}{|W|} J_{\{1\}}^L(\overline{f}_{Q,g}),$$
$$J_M(\gamma, \mathrm{Ad}(g^{-1})f) = \sum_{Q\in\mathcal{F}(M_0)} J_M^L(\gamma, \overline{f}_{Q,g}),$$

where

$$\overline{f}_{Q,g}(\ell) = \ell^{\rho_Q} \int_{\mathbf{K}} \int_{\mathbf{V}} f(k^{-1}\ell vk) u_Q'(k, y)\, dv\, dk$$
$$u_Q'(k, y) = \int_{\mathfrak{a}_L^G} \Gamma_Q^G(X, -H_0(ky))\, dX.$$

From this Arthur deduced [8, Cor. 8.3] that there exists a family of complex numbers $\{a^L(S,u) \mid L \in \mathcal{L}(M_0),\ u \in \mathcal{U}_L(F)\}$ such that

$$J_{\{1\}}(f) = \sum_{L\in\mathcal{L}(M_0)} \sum_{u\in\mathcal{U}_L(F)/\mathrm{Ad}(\mathbf{L}_S)} \frac{|W^L|}{|W|} a^L(S,u) J_L(u, f_S).$$

Using this (4.14) becomes

$$\begin{aligned} J_{\mathfrak{o}}(f) =& \iota^G(\sigma)^{-1} \int_{\mathbf{G}_{\sigma,S}\backslash\mathbf{G}_S} \sum_{L\in\mathcal{L}^{G_\sigma}(M_{1,\sigma})} \sum_{R\in\mathcal{F}^{G_\sigma}(L)} \frac{|W^L|}{|W^{G_\sigma}|} \\ &\times \sum_{u\in\mathcal{U}_L(F)/\mathrm{Ad}(\mathbf{L}_S)} a^L(S,u) J_L^{M_R}(u, \Phi_{R,y,T_1-T_{1,\sigma}})\, dy \end{aligned}$$

writing $\mathcal{L}_\sigma^0(M_1) := \{M \in \mathcal{L}(M_1) \mid \mathfrak{a}_M = \mathfrak{a}_{M_\sigma}\}$,

$$\begin{aligned} =& \iota^G(\sigma)^{-1} \sum_{M\in\mathcal{L}_\sigma^0(M_1)} \frac{|W^{M_\sigma}|}{|W^{G_\sigma}|} \sum_{u\in\mathcal{U}_{M_\sigma}(F)/\mathrm{Ad}(\mathbf{M}_{\sigma,S})} \\ & a^{M_\sigma}(S,u) |D^G(\sigma)|_{\mathbb{A}_S}^{1/2} \int_{\mathbf{G}_{\sigma,S}\backslash\mathbf{G}_S} \sum_{R\in\mathcal{F}^{G_\sigma}(M_\sigma)} J_{M_\sigma}^{M_R}(u, \Phi_{R,y,T_1-T_{1,\sigma}})\, dy. \end{aligned}$$

Going back from G_σ to G by Lem. 4.4, we obtain

$$J_{\mathfrak{o}}(f) = \iota^G(\sigma)^{-1} \sum_{M\in\mathcal{L}_\sigma^0(M_1)} \frac{|W^{M_\sigma}|}{|W^{G_\sigma}|} \sum_{u\in\mathcal{U}_{M_\sigma}(F)/\mathrm{Ad}(\mathbf{M}_{\sigma,S})} a^{M_\sigma}(S,u) J_M(\sigma u, f). \tag{4.15}$$

4.4 The fine $\mathfrak{O}$-expansion

Our final step is to get rid of σ from (4.15). We say that $\gamma, \gamma' \in M(F)$ is (M,S)-*equivalent* (notated as $\gamma \underset{(M,S)}{\sim} \gamma'$) if there exists $\delta \in M(F)$ such that

- $\gamma'_s = \delta\gamma_s\delta^{-1}$ and
- γ'_u and $\delta\gamma_u\delta^{-1}$ are conjugate in $\mathbf{M}_S$.

We write $(M(F) \cap \mathfrak{o})_{M,S}$ for the (finite) set of (M,S)-equivalence classes in $M(F) \cap \mathfrak{o}$. Noting that $\sigma \in M(F)$ is elliptic if and only if $M \in \mathcal{L}_\sigma^0(M_1)$, we set

$$a^M(S,\gamma) := \frac{\epsilon^M(\gamma_s)}{\iota^M(\gamma_s)} \sum_{\substack{u\in\mathcal{U}_{M_{\gamma_s}}(F)/\mathrm{Ad}(\mathbf{M}_{\gamma_s}) \\ \gamma_s u \underset{(M,S)}{\sim} \gamma}} a^{M_{\gamma_s}}(S,u),$$

where

$$\epsilon^M(\sigma) := \begin{cases} 1 & \text{if } \sigma \text{ is elliptic in } M(F), \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to write

$$J_{\mathfrak{o}}(f) = \sum_{M \in \mathcal{L}(M_1)} \frac{|W^M|}{|W|} \frac{|W| \cdot |W^{M_\sigma}| \cdot \iota^M(\sigma)}{|W^M| \cdot |W^{G_\sigma}| \cdot \iota^G(\sigma)} \sum_{\substack{\gamma \in (M(F) \cap \mathfrak{o})_{M,S} \\ \gamma_s = \sigma}} a^M(S,\gamma) J_M(\gamma, f)$$

$$= \sum_{M \in \mathcal{L}(M_0)} \frac{|W^M|}{|W|} \sum_{\substack{\mathfrak{o}_M \in \mathfrak{O}(M) \\ \mathrm{Ad}(G(F))\mathfrak{o}_M = \mathfrak{o}}} \sum_{\substack{\gamma \in (M(F) \cap \mathfrak{o})_{M,S} \\ \gamma_s = \sigma}} a^M(S,\gamma) J_M(\gamma, f)$$

$$= \sum_{M \in \mathcal{L}(M_0)} \frac{|W^M|}{|W|} \sum_{\gamma \in (M(F) \cap \mathfrak{o})_{M,S}} a^M(S,\gamma) J_M(\gamma, f).$$

Theorem 4.5 ([9] Th.9.2). *For each compact neighborhood $\Omega \subset \mathfrak{A}_G \backslash \mathbf{G}$ of 1, there exists a finite set of places S_Ω such that we have*

$$J(f) = \sum_{M \in \mathcal{L}(M_0)} \frac{|W^M|}{|W|} \sum_{\gamma \in M(F)_{M,S}} a^M(S,\gamma) J_M(\gamma, f)$$

for $S \supset S_\Omega$ and $f \in C_c^\infty(\mathbf{G}/\mathfrak{A}_G)$ supported on Ω.

We must remark that $a^M(S,\gamma)$ are constants which are not well-understood. The only information for these is the following theorem.

Theorem 4.6 ([9] Th. 8.2). *For a semisimple $\gamma \in G(F)$ and sufficiently large S, we have*

$$a^G(S,\gamma) = \begin{cases} \dfrac{\mathrm{meas}(G_\gamma(F)\mathfrak{A}_G \backslash \mathbf{G}_\gamma)}{\iota^G(\gamma)} & \text{if } \gamma \text{ is elliptic in } G(F), \\ 0 & \text{otherwise.} \end{cases}$$

5 Fine χ-expansion

Next we calculate the spectral terms. Let us recall the definition of $J_{\mathfrak{X}}^T(f)$. We have the induced space $\mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}}$ § 2.6. Define the linear transformation $\Omega_{\pi,\mathfrak{X}}^T(P,\lambda)$ on this space by

$$\int_{\mathbf{K}} \int_{M(F)\mathfrak{A}_M\backslash\mathbf{M}} [\Omega_{\pi,\mathfrak{X}}^T(P,\lambda)\phi](mk)\overline{\phi'(mk)}\, dm\, dk = \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} \wedge^T E_P(x,\phi_\lambda)\overline{\wedge^T E_P(x,\phi'_\lambda)}\, dx,$$

for any ϕ, $\phi' \in \mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}}$. From (2.6), we know that

$$\wedge_2^T K_{\mathfrak{X}}(x,y) = \wedge^T K_{\mathfrak{X}}(x,y) = \sum_{P \in \mathcal{F}(P_0)} \sum_{\pi \in \Pi(\mathbf{M}^1)} \frac{1}{|\mathcal{P}(M)|}$$

$$\int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi \in \mathfrak{B}_{\pi,\mathfrak{X}}} \wedge^T E_P(x, \mathcal{I}_P^G(\pi_\lambda, f)\varphi_\lambda)\overline{\wedge^T E_P(y,\varphi_\lambda)}\, d\lambda.$$

Replacing this in (3.6), we have

$$J_{\mathfrak{X}}^T(f) = \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \int_{i(\mathfrak{a}_M^G)^*} \frac{1}{|\mathcal{P}(M)|} \operatorname{tr}\big(\Omega_{\pi,\mathfrak{X}}^T(P,\lambda)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)\big)\, d\lambda, \tag{5.1}$$

where $\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda)$ denotes the representation of $\mathbf{G}$ on $\mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}}$. We would like to obtain a more explicit expression for this.

5.1 An application of the Paley-Wiener theorem

We put

$$\omega^T(\lambda,\lambda',\phi,\phi') := \sum_{Q=LV\in\mathcal{F}(P_0)} \sum_{w\in W_{M,L}} \sum_{w'\in W_{M',L}} \operatorname{meas}(\mathfrak{a}_L^G/\mathbb{Z}[\Delta_Q^\vee])$$
$$\times \frac{\langle M(w,\pi_\lambda)\phi, M(w',\pi'_{\lambda'})\phi'\rangle e^{(w(\lambda)-w'(\lambda'))(T)}}{\prod_{\alpha\in\Delta_Q}(w(\lambda)-w'(\lambda'))(\alpha^\vee)},$$

for $\phi\in\mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}}$, $\phi'\in\mathcal{A}_2(\mathbf{U}'M'(F)\mathfrak{A}_{M'}\backslash\mathbf{G})_{\pi',\mathfrak{X}}$, $\lambda\in i\mathfrak{a}_M^*$, $\lambda'\in i\mathfrak{a}_{M'}^*$. Extending the $GL(2)$ case, Langlands proved that if ϕ and ϕ' belong to the space of (induced from) cusp forms, the L^2-inner product of truncated Eisenstein series $\langle\Lambda^T E_P(\phi_\lambda), \Lambda^T E_{P'}(\phi'_{\lambda'})\rangle$ is given by $\omega^T(\lambda,\lambda',\phi,\phi')$ [30, § 9]. For general ϕ, ϕ' we have the following weaker result.

Proposition 5.1 ([5] Th. 9.1). *There exist $\epsilon>0$ and a locally bounded function $\rho(\lambda,\lambda')$ on $i\mathfrak{a}_M^*\times i\mathfrak{a}_{M'}^*$ such that*

$$|\langle\Lambda^T E_P(\phi_\lambda), \Lambda^T E_{P'}(\phi'_{\lambda'})\rangle - \omega^T(\lambda,\lambda',\phi,\phi')| \le \rho(\lambda,\lambda')\|\phi\|\cdot\|\phi'\|e^{-\epsilon\|T\|}.$$

Here the norms on the right are given by

$$\|\phi\|^2 := \int_{\mathbf{K}}\int_{M(F)\mathfrak{A}_M\backslash\mathbf{M}} |\phi(mk)|^2\, dm\, dk.$$

Example 5.2. *In the $GL(2)$ case, we write χ, χ' for π and $\pi'\in\Pi(\mathbf{T}^1)$. Then we have*

$$\begin{aligned}\langle\Lambda^T E_B(\phi_\lambda), \Lambda^T E_B(\phi'_{\lambda'})\rangle &= \langle E_B(\phi_\lambda), \Lambda^T E_B(\phi'_{\lambda'})\rangle\\ &= \int_{B(F)\mathfrak{A}_G\backslash\mathbf{G}} E_B(x,\phi_\lambda)\big(\overline{\phi'_{\lambda'}(x)} - \overline{E_B(x,\phi'_{\lambda'})}_B\widehat{\tau}_B(H_0(x)-T)\big)\, dx\\ &= \int_{\mathbf{U}T(F)\mathfrak{A}_G\backslash\mathbf{G}} E_B(x,\phi_\lambda)_B\big(\overline{\phi'_{\lambda'}(x)} - \overline{E_B(x,\phi'_{\lambda'})}_B\widehat{\tau}_B(H_0(x)-T)\big)\, dx\end{aligned}$$

using § 2.4 (1-ii),

$$= \int_{\mathbf{U}T(F)\mathfrak{A}_G\backslash\mathbf{G}} (\phi_\lambda(x) + M(w,\chi_\lambda)\phi_\lambda(x))$$
$$\times\, (\overline{\phi'_{\lambda'}(x)}(1-\widehat{\tau}_B(H_0(x)-T)) - \overline{M(w,\chi'_{\lambda'})\phi'_{\lambda'}(x)}\widehat{\tau}_B(H_0(x)-T))\, dx.$$

Using the Iwasawa decomposition this becomes, for $\mathrm{Re}\alpha^\vee(\lambda') > \pm\mathrm{Re}\alpha^\vee(\lambda)$,

$$\int_{-\infty}^{\alpha(T)} \langle\phi,\phi'\rangle e^{t\alpha^\vee(\lambda+\overline{\lambda}')/2} + \langle M(w,\chi_\lambda)\phi,\phi'\rangle e^{t\alpha^\vee(w(\lambda)+\overline{\lambda}')/2}\frac{dt}{\sqrt{2}}$$
$$- \int_{\alpha(T)}^{\infty} \langle\phi, M(w,\chi'_{\lambda'})\phi'\rangle e^{t\alpha^\vee(\lambda+w(\overline{\lambda}'))/2} + \langle M(w,\chi_\lambda)\phi, M(w,\chi'_{\lambda'})\phi'\rangle e^{t\alpha^\vee(w(\lambda+\overline{\lambda}'))/2}\frac{dt}{\sqrt{2}}$$
$$= \frac{\sqrt{2}e^{(\lambda+\overline{\lambda}')(T)}}{\alpha^\vee(\lambda+\overline{\lambda}')}\langle\phi,\phi'\rangle + \frac{\sqrt{2}e^{(w(\lambda)+\overline{\lambda}')(T)}}{\alpha^\vee(w(\lambda)+\overline{\lambda}')}\langle M(w,\chi_\lambda)\phi,\phi'\rangle$$
$$+ \frac{\sqrt{2}e^{(\lambda+w(\overline{\lambda}'))(T)}}{\alpha^\vee(\lambda+w(\overline{\lambda}'))}\langle\phi, M(w,\chi'_{\lambda'})\phi'\rangle + \frac{\sqrt{2}e^{w(\lambda+\overline{\lambda}')(T)}}{\alpha^\vee(w(\lambda+\overline{\lambda}'))}\langle M(w,\chi_\lambda)\phi, M(w,\chi'_{\lambda'})\phi'\rangle.$$

If we restrict the last formula to $(\lambda,\lambda') \in i\mathfrak{a}_T^*\times i\mathfrak{a}_T^*$, *we obtain* $\omega^T(\lambda,\lambda',\phi,\phi')$. *In the higher rank cases,* π *can be in the residual spectrum, an irreducible quotient of an induced from cuspidal representation. Since it throw away some submodules which might contribute to the inner product formula, we can control the inner product only asymptotically.*

We would like to replace $\omega^T(\lambda,\lambda',\phi,\phi')$ for the inner product of truncated Eisenstein series in (5.1). But this is not allowed because the domain of the integral is not compact. We bypass this difficulty in the following way.

Fix an $\mathbb{R}$-minimal parabolic subgroup $P_\infty = M_\infty U_\infty$ of $\mathbf{G}_\infty$ which is contained in the global minimal parabolic subgroup $\mathbf{P}_{0,\infty}$. Write $\mathfrak{a}_\infty$ for the Lie algebra of the $\mathbb{R}$-split component A_∞ of $Z(M_\infty)$. Choose a Cartan subalgebra $\mathfrak{h}_{\mathbf{K}_\infty}$ in the Lie algebra of $\mathbf{K}_\infty \cap M_\infty$ and set $\mathfrak{h} := i\mathfrak{h}_{\mathbf{K}_\infty} \oplus \mathfrak{a}_\infty$, a Cartan subalgebra of $\mathfrak{g}_\infty$. Write $W(\mathfrak{h}_\mathbb{C})$ for the Weyl group of $\mathfrak{h}_\mathbb{C}$ in $\mathfrak{g}_\infty(\mathbb{C})$ and let $\mathcal{E}(\mathfrak{h})^{W(\mathfrak{h}_\mathbb{C})}$ be the space of $W(\mathfrak{h}_\mathbb{C})$-invariant compactly supported distributions on $\mathfrak{h}$. The following is a corollary of the Paley-Wiener theorem for real reductive groups.

Proposition 5.3 (Multiplier theorem, [7] Th. 4.2). *For* $\gamma \in \mathcal{E}(\mathfrak{h})^{W(\mathfrak{h}_\mathbb{C})}$ *and* $f_\infty \in \mathcal{H}(\mathbf{G}_\infty)$, *we can find* $f_{\infty,\gamma} \in \mathcal{H}(\mathbf{G}_\infty)$ *such that*

$$\pi_\infty(f_{\infty,\gamma}) = \widehat{\gamma}(\chi_{\pi_\infty})\pi_\infty(f_\infty), \quad \forall\pi_\infty \in \Pi(\mathbf{G}_\infty).$$

Here $\widehat{\gamma}$ *denotes the Fourier transform of* γ *and* $\chi_{\pi_\infty} \in \mathfrak{h}_\mathbb{C}^*$ *is a representative of the infinitesimal character of* π_∞.

Write $\mathfrak{h}^G$ for the kernel of $\mathfrak{h} \overset{\exp}{\to} \mathbf{G}_\infty \overset{H_G}{\to} \mathfrak{a}_G$. Applying this proposition to the infinite component of $f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$, we have:

Corollary 5.4. *For* $\gamma \in \mathcal{E}(\mathfrak{h}^G)^{W(\mathfrak{h}_\mathbb{C})}$, *we have a linear map* $f \mapsto f_\gamma$ *on* $\mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$ *such that*

$$\pi(f_\gamma) = \widehat{\gamma}(\chi_{\pi_\infty})\pi(f), \quad \forall\pi \in \Pi(\mathbf{G}^1).$$

We use the abbreviation

$$\Psi^T_{\mathfrak{X},\pi}(\lambda, f) := \frac{1}{|\mathcal{P}(\mathcal{M})|}\mathrm{tr}(\Omega^T_{\pi,\mathfrak{X}}(P,\lambda)\mathcal{I}^G_{P,\mathfrak{X}}(\pi_\lambda, f)).$$

There exists $C_0 > 0$ such that if $\alpha(T) > C_0$, $\forall \alpha \in \Delta_0$ we have

$$\begin{aligned} J^T_{\mathfrak{X}}(f_\gamma) &= \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \int_{i(\mathfrak{a}_M^G)^*} \Psi^T_{\mathfrak{X},\pi}(\lambda, f_\gamma)\, d\lambda \\ &= \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \int_{i(\mathfrak{a}_M^G)^*} \widehat{\gamma}(\chi_{\pi_\infty}+\lambda)\Psi^T_{\mathfrak{X},\pi}(\lambda, f)\, d\lambda \\ &= \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \int_{i(\mathfrak{a}_M^G)^*} \Psi^T_{\mathfrak{X},\pi}(\lambda, f) \int_{\mathfrak{h}^G} \gamma(X) e^{(\chi_{\pi_\infty}+\lambda)(X)}\, dX\, d\lambda \\ &= \int_{\mathfrak{h}^G} \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \psi^T_{\mathfrak{X},\pi}(X, f)\gamma(X) e^{\chi_{\pi_\infty}(X)}\, dX, \end{aligned}$$

where

$$\psi^T_{\mathfrak{X},\pi}(X, f) := \int_{i(\mathfrak{a}_M^G)^*} \Psi^T_{\mathfrak{X},\pi}(\lambda, f) e^{\lambda(X)}\, d\lambda.$$

In particular, if we write $\gamma_H := |W(\mathfrak{h}_\mathbb{C})|^{-1} \sum_{w\in W(\mathfrak{h}_\mathbb{C})} \delta_{w^{-1}(H)}$ (average of Dirac distributions), then for T with $\alpha(T) > C_0$, $\forall \alpha \in \Delta_0$ we have

$$J^T_{\mathfrak{X}}(f_{\gamma_H}) = \frac{1}{|W(\mathfrak{h}_\mathbb{C})|} \sum_{w\in W(\mathfrak{h}_\mathbb{C})} \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \psi^T_{\mathfrak{X},\pi}(w^{-1}(H), f) e^{\chi_{\pi_\infty}(w^{-1}(H))}. \tag{5.2}$$

Let $p^T(H)$ be the right hand side. This is a polynomial in T and is smooth in H. We can recover $J^T_{\mathfrak{X}}(f)$ as $p^T(0) =$"$\delta_0(p^T)$".

The last expression is incorrect because the right hand side of (5.2) has non zero real exponent, i.e. is not tempered. Instead, we look at the coefficient

$$\psi^T_\lambda(H) := \frac{1}{|W(\mathfrak{h}_\mathbb{C})|} \sum_{P\in\mathcal{F}(P_0)} \sum_{\substack{(w,\pi)\in W(\mathfrak{h}_\mathbb{C})\times\Pi(\mathbf{M}^1) \\ w(\mathrm{Re}\chi_{\pi_\infty})=\lambda}} \psi^T_{\mathfrak{X},\pi}(w^{-1}(H), f) e^{\mathrm{Im}\chi_{\pi_\infty}(w^{-1}(H))}$$

of each real exponent λ of $p^T(H)$: $p^T(H) = \sum_\lambda \psi^T_\lambda(H) e^{\lambda(H)}$, the sum is finite. We can asymptotically approximate these by polynomial functions:

Lemma 5.5 ([6] I, Prop. 5.1). *There exists a unique (finite) family $\{p^T_\lambda(H)\}$ of polynomials in T which satisfies the following conditions for some C, $\epsilon > 0$. For any differential operator D on $\mathfrak{h}^G$, we can find $c_D > 0$ such that*

(1) $|D(\psi^T_\lambda(H) - p^T_\lambda(H))| \le c_D \exp(-\epsilon \inf_{\alpha\in\Delta_0} \alpha(T))(1 + \|T\|)^{d_0}$, for $H \in \mathfrak{h}^G$ and sufficiently positive T,

(2) $|Dp^T_\lambda(H)| \le c_D(1 + \|H\|)^{d_0}(1 + \|T\|)^{d_0}$, for $H \in \mathfrak{h}^G$ and $T \in \mathfrak{a}_0$.

Here d_0 is the maximum of the degrees of p^T_λ.

By construction, these $p^T_\lambda(H)$ are tempered and we can consider

$$p^T_\lambda(\beta) := \int_{\mathfrak{h}^G} p^T_\lambda(H)\beta(H)\, dH, \quad \beta \in \mathcal{S}(\mathfrak{h}^G),$$

a polynomial in T. Moreover we have

(1) For $\beta \in \mathcal{S}(\mathfrak{h}^G)$ satisfying

$$\int_{\mathfrak{h}^G} \beta(H)\, dH = 1,$$

we have $\lim_{\epsilon\to 0} p_\lambda^T(\beta_\epsilon) = p_\lambda^T(0)$ where $\beta_\epsilon(H) := \epsilon^{-\dim \mathfrak{h}^G}\beta(\epsilon^{-1}H)$.

(2) For $\beta \in \mathcal{S}(\mathfrak{h}^G)$, we have

$$\int_{\mathfrak{h}^G} \psi_\lambda^T(H)\beta(H)\, dH - p_\lambda^T(\beta) \to 0$$

as $\alpha(T) \to \infty$ for any $\alpha \in \Delta_0$.

Now we are ready to calculate $J_{\mathfrak{X}}^T(f)$. Take $B \in C_c^\infty(i(\mathfrak{h}^G)^*)$ and write B_M for the restriction of this to $i(\mathfrak{a}_M^G)^* \subset i(\mathfrak{h}^G)^*$. There is a $\beta \in \mathcal{S}(\mathfrak{h}^G)$ such that

$$B(\lambda) = \int_{\mathfrak{h}^G} \beta(H) e^{\lambda(H)}\, dH.$$

If we put $P^T(B) := \sum_\lambda p_\lambda^T(\beta)$, then the theory of Fourier transformation and the above (1), (2) imply:

(1) If $B(0) = 1$ we have $\lim_{\epsilon\to 0} P^T(B_\epsilon) = J_{\mathfrak{X}}^T(f)$, where $B_\epsilon(\lambda) := B(\epsilon\lambda)$.

(2) $P^T(B)$ is the unique polynomial in T such that

$$\sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \int_{i(\mathfrak{a}_M^G)^*} \Psi_{\mathfrak{X},\pi}^T(\lambda, f) B_M(\lambda)\, d\lambda - P^T(B)$$

goes to zero as $\alpha(T)$ tends to infinity for any $\alpha \in \Delta_0$.

Consider the linear transformation

$$A_{P,\pi,\mathfrak{X}}(\lambda,\lambda') := \sum_{Q\in\mathcal{F}(P_0)} \sum_{w,\,w'\in W_{M,L}} \frac{e^{(w'(\lambda')-w(\lambda))(T)}}{\theta_Q^G(w'(\lambda')-w(\lambda))} M(w,\pi_\lambda)^{-1} M(w',\pi'_\lambda)$$

on $\mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}}$. Since the global intertwining operators are unitary on the imaginary axis, we have

$$\begin{aligned}\langle A_{P,\pi,\mathfrak{X}}(\lambda,\lambda')\phi',\phi\rangle &= \sum_{Q\in\mathcal{F}(P_0)} \sum_{w,\,w'\in W_{M,L}} \frac{\langle M(w',\lambda')\phi', M(w,\lambda)\phi\rangle e^{(w'(\lambda')-w(\lambda))(T)}}{\theta_Q^G(w'(\lambda')-w(\lambda))}\\ &= \omega^T(\lambda',\lambda,\phi',\phi).\end{aligned}$$

It is shown that $\omega^T(\lambda',\lambda,\phi',\phi)$ is holomorphic on $i(\mathfrak{a}_M^G)^* \times i(\mathfrak{a}_M^G)^*$ [5, Cor .9.2], and we can define $\omega_{\mathfrak{X},\pi}^T(P,\lambda) := A_{P,\pi,\mathfrak{X}}(\lambda,\lambda)$ for $\lambda \in i(\mathfrak{a}_M^G)^*$. Since B_M are compactly supported, we can apply Prop. 5.1 to see that $P^T(B)$ is the unique polynomial in T such that

$$\sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \frac{1}{|\mathcal{P}(M)|} \int_{i(\mathfrak{a}_M^G)^*} \mathrm{tr}[\omega_{\mathfrak{X},\pi}^T(P,\lambda)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)] B_M(\lambda)\, d\lambda - P^T(B)$$

goes to zero as $\alpha(T)$ tends to infinity for any $\alpha \in \Delta_0$.

5.2 Logarithmic derivatives

Yet we need to give an explicit expression for $\omega^T_{\mathfrak{X},\pi}(P,\lambda)$. We first look at the simplest case.

Example 5.6. *Suppose $G = GL(2)$. In the notation of Ex. 5.2, we have*

$$\begin{aligned}
A_{B,\chi,\mathfrak{X}}(\lambda,\lambda') &= \frac{\sqrt{2}e^{(\lambda'-\lambda)(T)}}{\alpha^\vee(\lambda'-\lambda)} + \frac{\sqrt{2}e^{(w(\lambda')-\lambda)(T)}}{\alpha^\vee(w(\lambda')-\lambda)}M(w,\chi_{\lambda'}) \\
&+ \frac{\sqrt{2}e^{(\lambda'-w(\lambda))(T)}}{\alpha^\vee(\lambda'-w(\lambda))}M(w,\chi_\lambda)^{-1} + \frac{\sqrt{2}e^{w(\lambda'-\lambda)(T)}}{\alpha^\vee(w(\lambda'-\lambda))}M(w,\chi_\lambda)^{-1}M(w,\chi_{\lambda'}) \\
=& \frac{\sqrt{2}}{\alpha^\vee(\lambda'-\lambda)}\left(e^{(\lambda'-\lambda)(T)} - e^{(\lambda'-\lambda)(w(T))}M_{\overline{B}|B}(1,\chi_\lambda)^{-1}M_{\overline{B}|B}(1,\chi_{\lambda'})\right) \\
&+ \frac{\sqrt{2}}{\alpha^\vee(w(\lambda')-\lambda)}\left(e^{(w(\lambda')-\lambda)(T)}M(w,\chi_{\lambda'}) - e^{(w(\lambda')-\lambda)(w(T))}M_{\overline{B}|B}(1,\chi_\lambda)^{-1}M_{\overline{B}|B}(w,\chi_{\lambda'})\right),
\end{aligned}$$

where we have written $M_{\overline{B}|B}(1,\chi_\lambda) := w\circ M(w,\chi_\lambda)$ and $M_{\overline{B}|B}(w,\chi_\lambda) := w$. This illustrates how $A_{B,\chi,\mathfrak{X}}(\lambda,\lambda')$ becomes holomorphic on $\lambda=\lambda'$. Moreover, writing $\lambda' = \lambda + t\alpha/2$, the first row in the right hand side restricts to $\lambda=\lambda'$ as

$$\begin{aligned}
&\lim_{t\to 0}\frac{\sqrt{2}}{t}\left[e^{\alpha(T)t/2} - e^{-\alpha(T)t/2}M_{\overline{B}|B}(1,\chi_\lambda)^{-1}M_{\overline{B}|B}(1,\chi_{\lambda+t\alpha/2})\right] \\
&= \lim_{t\to 0}\frac{\sqrt{2}}{t}\left[1 - e^{-\alpha(T)t}M_{\overline{B}|B}(1,\chi_\lambda)^{-1}M_{\overline{B}|B}(1,\chi_{\lambda+t\alpha/2})\right],
\end{aligned}$$

which gives rise to the logarithmic derivative term in the trace formula of $GL(2)$.

To produce the higher dimensional analogue of the logarithmic derivative, we use two kinds of (G,M)-families. We need the following recapitulation of intertwining operators.

Take M, $M'\in\mathcal{L}(M_0)$. For $P\in\mathcal{P}(M)$, $P'\in\mathcal{P}(M')$ and $w\in W_{M,M'}$, we define

$$[M_{P'|P}(w,\pi_\lambda)\phi](x) = \int_{(\mathbf{U}'\cap^w\mathbf{U})\backslash\mathbf{U}'}\phi(w^{-1}ux)e^{\langle\lambda+\rho_P,H_P(w^{-1}ux)\rangle}\,du\cdot e^{\langle-(w(\lambda)+\rho_{P'}),H_{P'}(x)\rangle},$$

a linear operator $\mathcal{A}_2(\mathbf{U}M(F)\mathfrak{A}_M\backslash\mathbf{G})_{\pi,\mathfrak{X}} \to \mathcal{A}_2(\mathbf{U}'M'(F)\mathfrak{A}_{M'}\backslash\mathbf{G})_{w(\pi),\mathfrak{X}}$. One can easily show that

$$\begin{aligned}
v\circ M_{P'|P}(w,\pi_\lambda) &= e^{-\langle(w(\lambda)+\rho_{P'}),T_1-v^{-1}(T_1)\rangle}M_{v(P')|P}(vw,\pi_\lambda) \\
M_{P'|P}(w,\pi_\lambda)\circ v^{-1} &= e^{\langle\lambda+\rho_P,T_1-v^{-1}(T_1)\rangle}M_{P'|v(P)}(wv^{-1},v(\pi_\lambda)).
\end{aligned}$$

If we choose P_1, $P_1'\in\mathcal{F}(P_0)$ and v_1, $v_1'\in W$ such that $P={}^{v_1}P_1$, $P'={}^{v_1'}P_1'$, we may write $w=v_1'w_1v_1^{-1}$ for some $w_1\in W_{M_1,M_1'}$. Then the above formulae imply that

$$\begin{aligned}
&M_{P'|P}(w,\pi_\lambda) \\
&= e^{\langle w_1v_1^{-1}(\lambda)+\rho_{P_1'},T_1-v_1'^{-1}(T_1)\rangle}e^{-\langle v_1^{-1}(\lambda)+\rho_{P_1},T_1-v_1^{-1}(T_1)\rangle}v_1'\circ M_{P_1'|P_1}(w_1,v_1^{-1}(\lambda))\circ v_1^{-1}.
\end{aligned} \tag{5.3}$$

This relation allows us to translate the basic properties of $M(w,\pi_\lambda)$ to those of $M_{P'|P}(w,\pi_\lambda)$. In particular the latter operator absolutely converges if the real part of λ belongs to a cone, and extends meromorphically to $\mathfrak{a}^*_{M,\mathbb{C}}$. This defines an intertwining operator at λ where it is holomorphic. Finally this fits into the functional equations:

$$E_{P'}(x, M_{P'|P}(w,\pi_\lambda)\phi_\lambda) = E_P(x,\phi_\lambda), \tag{5.4}$$

$$M_{P''|P'}(w', w(\pi_\lambda))M_{P'|P}(w,\pi_\lambda) = M_{P''|P}(w'w,\pi_\lambda), \quad P'' \in \mathcal{P}(M''),\ w' \in W_{M',M''}. \tag{5.5}$$

These results allow us to write $A_{P,\pi,\mathfrak{X}}(\lambda,\lambda')$ as:

$$\begin{aligned} A_{P,\pi,\mathfrak{X}}(\lambda,\lambda') &= \sum_{P_1 \supset P_0} \sum_{w,\, w' \in W_{M,M_1}} \frac{e^{(w'(\lambda') - w(\lambda))(T)}}{\theta^G_{P_1}(w'(\lambda') - w(\lambda))} M_{P_1|P}(w,\pi_\lambda)^{-1} M_{P_1|P}(w',\pi'_\lambda) \\ &= \sum_{P_1 \supset P_0} \sum_{\substack{v \in W_{M,M_1} \\ w \in W_{M,M}}} \frac{e^{v(w(\lambda') - \lambda)(T)}}{\theta^G_{P_1}(v(w(\lambda') - \lambda))} M_{P_1|P}(v,\pi_\lambda)^{-1} M_{P_1|P}(vw,\pi_\lambda) \end{aligned}$$

using (5.5)

$$\begin{aligned} &= \sum_{P_1 \supset P_0} \sum_{\substack{v \in W_{M,M_1} \\ w \in W_{M,M}}} \frac{e^{v(w(\lambda') - \lambda)(T)} e^{(w(\lambda') - \lambda)(T_1 - v^{-1}(T_1))}}{\theta^G_{v^{-1}(P_1)}(w(\lambda') - \lambda)} \\ &\qquad \times M_{v^{-1}(P_1)|P}(1,\pi_\lambda)^{-1} M_{v^{-1}(P_1)|P}(w,\pi'_\lambda) \end{aligned}$$

putting $Q := v^{-1}(P_1) \in \mathcal{P}(M)$

$$= \sum_{Q \in \mathcal{P}(M)} \sum_{w \in W_{M,M}} \frac{e^{\langle w(\lambda') - \lambda, Y_Q(T)\rangle}}{\theta^G_Q(w(\lambda') - \lambda)} M_{Q|P}(1,\pi_\lambda)^{-1} M_{Q|P}(w,\pi'_\lambda).$$

Here we have written $Y_Q(T) := T_1 + v^{-1}(T - T_1)$ for $Q = v^{-1}(P_1)$, $v \in W_{M,M_1}$. To calculate $\mathrm{tr}[\omega^T_{\pi,\mathfrak{X}}(P,\lambda)\mathcal{I}^G_{P,\mathfrak{X}}(\pi_\lambda,f)]$, which equals the restriction of

$$\sum_{w \in W_{M,M}} \sum_{Q \in \mathcal{P}(M)} \frac{e^{(w(\lambda') - \lambda)(Y_Q(T))}}{\theta^G_Q(w(\lambda') - \lambda)} \mathrm{tr}(M_{Q|P}(1,\pi_\lambda)^{-1} M_{Q|P}(w,\pi_{\lambda'})\mathcal{I}^G_{P,\mathfrak{X}}(\pi_\lambda,f)) \tag{5.6}$$

to $\lambda = \lambda'$, we note that

$$c_Q(T,\Lambda) := e^{\Lambda(Y_Q(T))}, \quad d^w_Q(\Lambda) := \mathrm{tr}(M_{Q|P}(1,\pi_\lambda)^{-1} M_{Q|P}(w,\pi_{\lambda+\Lambda})\mathcal{I}^G_{P,\mathfrak{X}}(\pi_\lambda,f))$$

form (G,M)-families. Then we know from Lem. 4.3 that (5.6) is smooth at any (λ,λ') and

$$\mathrm{tr}[\omega^T_{\pi,\mathfrak{X}}(P,\lambda)\mathcal{I}^G_{P,\mathfrak{X}}(\pi_\lambda,f)] = \sum_{w \in W_{M,M}} \sum_{P_1 \in \mathcal{F}(M)} c^{P_1}_M(T, w(\lambda) - \lambda) d^w_{P_1}{}'(\lambda_{L_w}, w(\lambda) - \lambda).$$

Here $L_w \in \mathcal{L}(M)$ is such that $\mathfrak{a}_{L_w} = \{H \in \mathfrak{a}_M \mid w(H) = H\}$ and

$$d_Q^w(\lambda_{L_w}, \Lambda) := \mathrm{tr}(M_{Q|P}(1, \pi_\lambda)^{-1} M_{Q|P}(w, \pi_{\lambda+\zeta}) \mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)),$$

if $\Lambda = w(\lambda) - \lambda + \zeta$, $\zeta \in i\mathfrak{a}_M^*$. (Notice that λ and ζ can be recovered from Λ and λ_{L_w}.)

We now combine the above with the result of § 5.1 to see that

$$\sum_{P \in \mathcal{F}(P_0)} \sum_{\pi \in \Pi(\mathbf{M}^1)} \frac{1}{|\mathcal{P}(M)|} \sum_{w \in W_{M,M}} \int_{i(\mathfrak{a}_M^G)^*} \sum_{P_1 \in \mathcal{F}(M)} c_M^{P_1}(T, w(\lambda) - \lambda) d_{P_1}^w{}'(\lambda_{L_w}, w(\lambda) - \lambda) B_M(\lambda)\, d\lambda \tag{5.7}$$

is asymptotic to $P^T(B)$. Write $\chi_M^{P_1}(T, \bullet)$ for the characteristic function of the convex hull of $\{Y_Q(T) \mid Q \in \mathcal{P}(M),\ Q \subset P_1\}$, then $c_M^{P_1}(T)$ is its Fourier transform [4, § 6]:

$$c_M^{P_1}(T, \mu) = \int_{(Y_Q(T))_{M_1} + \mathfrak{a}_M^{M_1}} \chi_M^{P_1}(T, X) e^{\mu(X)}\, dX.$$

Then the integral over $i(\mathfrak{a}_M^G)^*$ can be calculated as

$$\frac{1}{|\det(w - 1|\mathfrak{a}_M^{L_w})|} \sum_{P_1 \in \mathcal{F}(M)} \int_{(Y_Q(T))_{M_1} + \mathfrak{a}_M^{M_1}} \chi_M^{P_1}(T, H) \int_{i(\mathfrak{a}_M^{L_w})^*} \int_{i(\mathfrak{a}_{L_w}^G)^*} e^{\mu(H)} d_{P_1}^w{}'(\lambda, \mu) B_M((w-1)^{-1}(\mu) + \lambda)\, d\lambda\, d\mu\, dH$$

In this, the terms associated to $P_1 \not\supset L_w$ goes to 0 as $\alpha(T) \to \infty$, $\forall \alpha \in \Delta_0$. Those associated to $P_1 \supset L_w$ becomes

$$\frac{1}{|\det(w - 1|\mathfrak{a}_M^{L_w})|} c_{L_w}^{P_1}(T, 0) \int_{i(\mathfrak{a}_{L_w}^G)^*} d_{P_1}^w{}'(\lambda, 0) B_M(\lambda)\, d\lambda.$$

We conclude that (5.7) equals

$$\sum_{P \in \mathcal{F}(P_0)} \sum_{\pi \in \Pi(\mathbf{M}^1)} \frac{1}{|\mathcal{P}(M)|} \sum_{w \in W_{M,M}} \frac{1}{|\det(w - 1|\mathfrak{a}_M^{L_w})|} \int_{i(\mathfrak{a}_{L_w}^G)^*} \sum_{P_1 \in \mathcal{F}(L_w)} c_{L_w}^{P_1}(T, 0) d_{P_1}^w{}'(\lambda, 0) B_M(\lambda)\, d\lambda. \tag{5.8}$$

Final step towards the fine $\mathfrak{X}$-expansion is to look at the nature of the "logarithmic derivative"

$$\sum_{P_1 \in \mathcal{F}(L_w)} c_{L_w}^{P_1}(T, 0) d_{P_1}^w{}'(\lambda, 0) = \sum_{Q \in \mathcal{P}(L_w)} \left. \frac{c_Q(T, \Lambda) d_Q^w(\lambda, \Lambda)}{\theta_Q^G(\Lambda)} \right|_{\Lambda = 0}. \tag{5.9}$$

As in Ex. 5.6, we divide $d_Q^w(\lambda, \Lambda)$ as

$$d_Q^w(\lambda, \Lambda) = \mathrm{tr}[\big(M_{Q|P}(1, \pi_\lambda)^{-1} M_{Q|P}(1, \pi_{\lambda+\Lambda})\big) \circ \big(M_{P|P}(w, \pi_{\lambda+\Lambda}) \mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)\big)]$$

The latter simply restricts to $\Lambda = 0$. But in the former,

$$\mathcal{M}_Q(P,\pi,\lambda,\Lambda) := M_{Q|P}(1,\pi_\lambda)^{-1} M_{Q|P}(1,\pi_{\lambda+\Lambda}), \quad Q \in \mathcal{P}(M)$$

is a (G,M)-family in Λ and so is

$$\mathcal{M}_Q^T(P,\pi,\lambda,\Lambda) := c_Q(T,\lambda)\mathcal{M}_Q(P,\pi,\lambda,\Lambda).$$

This combined with Lem. 4.3 justifies to write the specialization of (5.9) to $\Lambda = 0$ as

$$\mathrm{tr}\Big[\Big(\sum_{Q\in\mathcal{P}(L_w)} \frac{1}{\theta_Q^G(\Lambda)} \mathcal{M}_Q^T(P,\pi,\lambda,\Lambda)\Big|_{\Lambda=0}\Big) M_{P|P}(w,\pi)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda,f)\Big].$$

Note that $M_{P|P}(w,\pi_{\Lambda+\lambda}) = M_{P|P}(w,\pi)$ for $\Lambda+\lambda \in i\mathfrak{a}_L^*$. Since this is a linear combination of certain derivatives at $\Lambda = 0$ of the exponential functions $c_Q(T,\Lambda) = e^{\Lambda(Y_Q(T))}$, the exponents being linear in T, this is a polynomial function in T. Since (5.8) and $P^T(B)$ are both polynomials and asymptotic to each other, they must coincide:

$$\begin{aligned} P^T(B) = &\sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \sum_{L\in\mathcal{L}(M)} \sum_{w\in W_{M,M}^{L,\mathrm{reg}}} \frac{1}{|\mathcal{P}(M)|} \frac{1}{|\det(w-1|\mathfrak{a}_M^L)|} \\ &\times \int_{i(\mathfrak{a}_L^G)^*} \mathrm{tr}(\mathcal{M}_L^T(P,\pi,\lambda,0) M_{P|P}(w,\pi)\mathcal{I}_{P,\mathfrak{X}}(\pi_\lambda,f)) B_M(\lambda)\,d\lambda. \end{aligned} \tag{5.10}$$

Here $W_{M,M}^{L,\mathrm{reg}} := \{w \in W_{M,M}^L \mid \det(w-1|\mathfrak{a}_M^L) \neq 0\}$. Once we have an equality, we can specialize it to $T = T_1$. Since $Y_Q(T_1) = T_1$, we have

$$\mathcal{M}_L^{T_1}(P,\pi,\lambda,0) = e^{\Lambda(T_1)}\mathcal{M}_L(P,\pi,\lambda,\Lambda)|_{\Lambda=0} = \mathcal{M}_L(P,\pi,\lambda,0).$$

Using p. 30 (1), we conclude from (5.10) that

$$\begin{aligned} J_{\mathfrak{X}}(f) = &\lim_{\epsilon\to 0} \sum_{P\in\mathcal{F}(P_0)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \sum_{L\in\mathcal{L}(M)} \sum_{w\in W_{M,M}^{L,\mathrm{reg}}} \frac{1}{|\mathcal{P}(M)|} \frac{1}{|\det(w-1|\mathfrak{a}_M^L)|} \\ &\times \int_{i(\mathfrak{a}_L^G)^*} \mathrm{tr}(\mathcal{M}_L(P,\pi,\lambda,0) M_{P|P}(w,\pi)\mathcal{I}_{P,\mathfrak{X}}(\pi_\lambda,f))(B_\epsilon)_M(\lambda)\,d\lambda \\ = &\lim_{\epsilon\to 0} \sum_{M\in\mathcal{L}(M_0)} \sum_{L\in\mathcal{L}(M)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \sum_{w\in W_{M,M}^{L,\mathrm{reg}}} \frac{|W^M|}{|W|} \frac{1}{|\det(w-1|\mathfrak{a}_M^L)|} \\ &\times \int_{i(\mathfrak{a}_L^G)^*} \frac{1}{|\mathcal{P}(M)|} \sum_{P\in\mathcal{P}(M)} \mathrm{tr}(\mathcal{M}_L(P,\pi,\lambda,0) M_{P|P}(w,\pi)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda,f))(B_\epsilon)_M(\lambda)\,d\lambda. \end{aligned} \tag{5.11}$$

Here $B \in C_c^\infty(i(\mathfrak{h}^G)^*)$ is such that $B(0) = 1$.

5.3 Normalization and estimation of intertwining operators

We hope to get rid of the limit $\epsilon \to 0$ and the factor B_ϵ from (5.11). For this we have to estimate the integrand and show that it converges without the factor B_ϵ. For such an estimation, we adopt the usual approach of normalizing intertwining operators.

The normalization is constructed locally. Consider a connected reductive group G over a local field F of characteristic zero. We adopt the local analogue of various notation presented above. In particular we write $\mathbf{G} = G(F)$ and fix its maximal compact subgroup $\mathbf{K}$ so that the Iwasawa decomposition $\mathbf{G} = \mathbf{PK}$ holds for any $P \in \mathcal{F}(M_0)$. For $M \in \mathcal{L}(M_0)$, write $\Pi_{\mathrm{adm}}(\mathbf{M})$ for the set of isomorphic classes of irreducible admissible representations (irreducible $(\mathfrak{m}_{\mathbb{C}}, \mathbf{K}^M)$-modules if F is archimedean) of $\mathbf{M}$. For $\pi \in \Pi(\mathbf{M})$ and $P \in \mathcal{P}(M)$, write $\mathcal{V}_P(\pi)$ for the space of smooth right $\mathbf{K}$-finite functions on $\mathbf{G}$ satisfying

$$\phi(umg) = \pi(m)\phi(g), \quad u \in \mathbf{U},\, m \in \mathbf{M},\, g \in \mathbf{G}.$$

The parabolically induced representation $\mathcal{I}_P^G(\pi_\lambda)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ is defined by

$$[\mathcal{I}_P^G(\pi_\lambda, g)\phi](x) := \phi(xg)e^{\langle \lambda+\rho_P, H_P(xg)\rangle}e^{-\langle \lambda+\rho_P, H_P(x)\rangle}, \quad g \in \mathbf{G},\, \phi(x) \in \mathcal{V}_P(\pi).$$

This is isomorphic to the usual parabolically induced representation by

$$\mathcal{I}_P^G(\pi_\lambda) \ni \phi(x) \longmapsto \phi(x)e^{\langle \lambda+\rho_P, H_P(x)\rangle} \in \mathrm{ind}_{\mathbf{P}}^{\mathbf{G}}[\pi_\lambda \otimes \mathbf{1}_{\mathbf{U}}].$$

As in the global case, we define the intertwining integral $M_{P'|P}(w, \pi_\lambda)$, ($P = MU$, $P' = M'U' \in \mathcal{F}(M_0)$, $w \in W_{M,M'}$) by

$$[M_{P'|P}(w,\pi_\lambda)\phi](x) := \int_{(\mathbf{U}' \cap {}^w\mathbf{U})\backslash \mathbf{U}'} \phi(w^{-1}ux)e^{\langle \lambda+\rho_P, H_P(w^{-1}ux)\rangle}\, du \cdot e^{\langle -w(\lambda)+\rho_{P'}), H_{P'}(x)\rangle}.$$

Proposition 5.7 ([27], [39] § 2.2). *(i)* $[M_{P'|P}(w,\pi_\lambda)\phi](x)$ *converges absolutely if* $\mathrm{Re}(\lambda)$ *belongs to some open cone in* $\mathfrak{a}_M^*$, *and meromorphically continued to the whole* $\mathfrak{a}_{M,\mathbb{C}}^*$.
(ii) If we write $\ell_P(w)$ *for the length function on* $W_M(G)$ *with respect to* $P \in \mathcal{P}(M)$ *[33, I.1.7], then for* $w \in W_{M,M'}$ *and* $w' \in W_{M',M''}$ *with* $\ell_P(w'w) = \ell_P(w) + \ell_{P'}(w')$, *the functional equation*

$$M_{P''|P'}(w', w(\pi_\lambda))M_{P'|P}(w,\pi_\lambda) = M_{P''|P}(w'w, \pi_\lambda)$$

holds.

Consider first the following two special cases.

(1) F is archimedean.

(2) G is quasisplit over F and π is generic with respect to some non-degenerate character of a maximal unipotent subgroup of $\mathbf{M}$.

(Recall that a *splitting* of G is a triple of a Borel subgroup B, a maximal torus T in B and a system of root vectors $\{X\} \subset \mathfrak{g}$ for the simple roots of T in B. G is *quasisplit* if it admits a splitting $\mathbf{spl}_G = (B, T, \{X\})$ which is stable under $\mathrm{Gal}(\overline{F}/F)$. Then a character θ of the unipotent radical $\mathbf{N}$ of $\mathbf{B}$ is *non-degenerate* if its stabilizer in $\mathbf{B}$ equals the center of $\mathbf{G}$. For such θ, a *θ-Whittaker functional* on $\pi \in \Pi(\mathbf{G})$ is a linear functional $\Lambda_\pi^\theta : V_\pi \longrightarrow \mathbb{C}$ on a realization of π such that

$$\Lambda_\pi^\theta(\pi(n)\xi) = \theta(n)\Lambda_\pi^\theta(\xi), \quad n \in \mathbf{N},\, \xi \in V_\pi.$$

We say that π is *θ-generic* if it admits a non-trivial θ-Whittaker functional.) In these cases, we have the automorphic L and ε-factors $L(s,\pi,r)$ and $\varepsilon(s,\pi,r,\psi)$ of π attached to certain finite dimensional continuous representation r of the L-group ${}^LG$ of G [17], ψ being a non-trivial character of F. (N.B. The L-group should be the Weil form ${}^LG = \widehat{G} \rtimes_{\rho_G} W_F$ instead of the Galois form $\widehat{G} \rtimes_{\rho_G} \mathrm{Gal}(\overline{F}/F)$ adopted in [17], since some important cocycles on $\mathrm{Gal}(\overline{F}/F)$ does not split while its inflation to W_F does.) In the case (1), these are defined in terms of the local Langlands correspondence established in [32]. In the case (2), the definition is given in [40, § 7]. Now let P, P', $\pi \in \Pi_{\mathrm{adm}}(\mathbf{M}^1)$, $\lambda \in i\mathfrak{a}_M^*$ be as above. Writing $\widehat{\mathfrak{u}}_w := \mathfrak{u}/{}^{w^{-1}}\mathfrak{u}' \cap \mathfrak{u}$, set

$$r_w : {}^LM \ni m \rtimes w \longmapsto \mathrm{Ad}(m) \circ \rho_G(w)|_{\widehat{\mathfrak{u}}_w} \in GL(\widehat{\mathfrak{u}}_w).$$

Define the normalization factor for $M_{P'|P}(w,\pi_\lambda)$ by

$$r_{P'|P}(w,\pi_\lambda,\psi) := \frac{L(0,\pi_\lambda,r_w)}{\varepsilon(0,\pi_\lambda,r_w,\psi)L(1,\pi_\lambda,r_w)}.$$

The normalized operator $N_{P'|P}(w,\pi_\lambda) := r_{P'|P}(w,\pi_\lambda,\psi)^{-1}M_{P'|P}(w,\pi_\lambda)$ enjoys the following properties [13, I, §§ 2,3], [40, Th. 7.9]:

(N1) $N_{P'|P}(w,\pi_\lambda)\mathcal{I}_P^G(\pi_\lambda,f) = \mathcal{I}_{P'}^G(w(\pi_\lambda),f)N_{P'|P}(w,\pi_\lambda)$, $f \in \mathcal{H}(\mathbf{G})$.

(N2) Without any length condition, the functional equation

$$N_{P''|P'}(w', w(\pi_\lambda))N_{P'|P}(w,\pi_\lambda) = N_{P''|P}(w'w,\pi_\lambda)$$

holds.

(N3) For $\lambda \in i\mathfrak{a}_M^*$, $N_{P'|P}(w,\pi_\lambda)$ is unitary.

(N4) In the case (1), $N_{P'|P}(w,\pi_\lambda)$ is a rational function in $(\alpha^\vee(\lambda))_{\alpha\in\Delta_P}$. In the case (2) it is a rational function in $(q_F^{-\alpha^\vee(\lambda)})_{\alpha\in\Delta_P}$, where q_F is the cardinality of the residue field of F.

(N5) In the case (1), and if π is tempered, then $r_{P'|P}(w,\pi_\lambda,\psi)$ has no poles in the region $\mathrm{Re}(\alpha^\vee(\lambda)) > 0$, $\forall\alpha \in \Delta_P$.

(N6) If G is unramified in case (2) and $\mathcal{I}_P^G(\pi_\lambda)$ admits a fixed vector ϕ^0 under the hyperspecial maximal compact subgroup $\mathbf{K}$, then $N_{P'|P}(w,\pi_\lambda)\phi^0(k) = \phi^0(k)$, $k \in \mathbf{K}$.

The property (N5) should also hold in the case (2) [40, Conj. 7.1].

Example 5.8. *Consider the case of $GL(2)$. If $\chi_\lambda = \chi_1|\ |_F^{\lambda_1} \otimes \chi_2|\ |_F^{\lambda_2}$, then the normalization factor equals*

$$r_{B|B}(w, \pi_\lambda, \psi) = r_{\overline{B}|B}(1, \pi_\lambda, \psi) = \frac{L(\lambda_1 - \lambda_2, \chi_1\chi_2^{-1})}{\varepsilon(\lambda_1 - \lambda_2, \chi_1\chi_2^{-1}, \psi)L(1 + \lambda_1 - \lambda_2, \chi_1\chi_2^{-1})}.$$

(N5) is clearly valid in this case.

In the general case, a normalization factor satisfying (N1) to (N6) was constructed by Langlands using the Plancherel measure [20, Lect.15]. We still use the notation $r_{P'|P}(w, \pi_\lambda, \psi)$ for this normalization factor, since these two essentially coincide in the above special cases. To obtain an expression of $r_{P'|P}(w, \pi_\lambda, \psi)$ in terms of L and ϵ-factors as illustrated above is also important in the arithmetic applications.

Going back to the global setting, we define

$$r_{P'|P}(w, \pi_\lambda) := \prod_v r_{P'|P}(w, \pi_{v,\lambda}, \psi_v), \quad N_{P'|P}(w, \pi_\lambda) := \bigotimes_v N_{P'|P}(w, \pi_{v,\lambda}).$$

Here $\psi = \bigotimes_v \psi_v$ is a non-trivial character of $\mathbb{A}/F$ and $\pi = \bigotimes_v \pi_v$. We use these to estimate

$$\sum_{\pi\in\Pi(\mathbf{M}^1)} \int_{i(\mathfrak{a}_L^G)^*} \|\mathcal{M}_L(P, \pi, \lambda, 0)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)\| \, d\lambda,$$

where $\|\ \|$ is the trace class norm. Define two (G, M)-families

$$\mathcal{N}_Q(P, \pi, \lambda, \Lambda) := N_{Q|P}(1, \pi_\lambda)^{-1} N_{Q|P}(1, \pi_{\lambda+\Lambda}), \quad r_Q(P, \pi, \lambda, \Lambda) := \frac{r_{Q|P}(1, \pi_{\lambda+\Lambda})}{r_{Q|P}(1, \pi_\lambda)}.$$

We apply Lem. 4.3 to $\mathcal{M}_Q(P, \pi, \lambda, \Lambda) = r_Q(P, \pi, \lambda, \Lambda)\mathcal{N}_Q(P, \pi, \lambda, \Lambda)$ to have

$$\mathcal{M}_L(P, \pi, \lambda, 0)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f) = \sum_{P_1\in\mathcal{F}(L)} r_L^{P_1}(P, \pi, \lambda, 0)\mathcal{N}'_{P_1}(P, \pi, \lambda, 0)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f).$$

Since $\mathcal{N}'_{P_1}(P, \pi, \lambda, 0)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)$ is rapidly decreasing in λ, it suffices to show that

$$\int_{i(\mathfrak{a}_L^G)^*} |r_L^{P_1}(P, \pi, \lambda, 0)|(1 + \|\lambda\|)^{-N} \, d\lambda$$

converges absolutely for sufficiently large N. Once we are reduced to the estimation of such a scalar valued function, we can deduce it from that of the inner product of two truncated Eisenstein series (using Prop. 5.1) . This is done in [6, II, § 9]. Finally we have the following.

Theorem 5.9 (The fine $\mathfrak{X}$-expansion). *For $f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$, we have*

$$J_{\mathfrak{X}}(f) = \sum_{M\in\mathcal{L}(M_0)} \sum_{L\in\mathcal{L}(M)} \sum_{\pi\in\Pi(\mathbf{M}^1)} \sum_{w\in W_{M,M}^{L,\mathrm{reg}}} \frac{|W^M|}{|W|} \frac{1}{|\det(w - 1|\mathfrak{a}_M^L)|}$$
$$\times \int_{i(\mathfrak{a}_L^G)^*} \frac{1}{|\mathcal{P}(M)|} \sum_{P\in\mathcal{P}(M)} \mathrm{tr}(\mathcal{M}_L(P, \pi, \lambda, 0)M_{P|P}(w, \pi)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda, f)) \, d\lambda.$$

5.4 Weighted characters

To obtain an expression of $J_{\mathfrak{X}}(f)$ analogous to Th. 4.5, we need weighted characters, the spectral counter part of weighted orbital integrals [13].

Let S be a finite set of places of F. Consider the local analogue

$$\mathcal{N}_Q(P,\pi,\lambda,\Lambda) = N_{Q|P}(1,\pi_\lambda)^{-1}N_{Q|P}(1,\pi_{\lambda+\Lambda}), \quad \pi\in\Pi_{\mathrm{adm}}(\mathbf{M}_S),\ \Lambda\in i\mathfrak{a}_M^*$$

of the (G,M)-family defined above. Since the singularity of $N_{Q|P}(1,\pi_\lambda)$ as a function in λ is isolated, its "logarithmic derivative" at λ

$$\mathcal{N}_M(P,\pi,\lambda) := \lim_{\Lambda\to 0}\sum_{Q\in\mathcal{P}(M)}\frac{1}{\theta_Q^G(\Lambda)}\mathcal{N}_Q(P,\pi,\lambda,\Lambda)$$

can be defined. This is meromorphic in λ whose singularity set is a locally finite union of affine hyperplanes whose vector parts are the zeros of coroots. Define the *weighted character* to be

$$J_M(\pi_\lambda,f) := \mathrm{tr}(\mathcal{N}_M(P,\pi,\lambda)\mathcal{I}_P^G(\pi_\lambda,f)).$$

More generally, for $\tau\in\Pi_{\mathrm{adm}}(\mathbf{L})$ ($L\subset M$), we define $J_M(\tau_\lambda,f) := J_M(\mathcal{I}_Q^M(\tau)_\lambda,f)$ with any $Q\in\mathcal{P}^M(L)$. This again is a meromorphic function having the same type singularities as $\mathcal{N}_M(P,\pi,\lambda)$ has. Note that, by taking trace, this is independent of $P\in\mathcal{P}(M)$ as the notation suggests. We also need the distribution

$$\mathcal{J}_M(\pi,X,f) := \int_{i\mathfrak{a}_{M,S}^{G*}} J_M(\pi_\lambda,f)e^{-\lambda(X)}\,d\lambda, \quad f\in\mathcal{H}(\mathbf{G}_S).$$

Here $\mathfrak{a}_{M,S} := H_M(\mathbf{M}_S)$ is $\mathfrak{a}_M$ itself or a lattice in $\mathfrak{a}_M$ (according to either S contains an archimedean place or not), and we have written $\mathfrak{a}_{M,S}^*$ for $\mathfrak{a}_M^*$ or $\mathfrak{a}_M^*/(\mathfrak{a}_{M,S}^\vee)$ accordingly. $\mathfrak{a}_{M,S}^\vee$ denotes the dual lattice of $\mathfrak{a}_{M,S}$ in the latter.

We look at the discrete part (i.e. the term associated to $L=G$) of the fine $\mathfrak{X}$-expansion Th. 5.9:

$$\sum_{M\in\mathcal{L}(M_0)}\sum_{\pi\in\Pi(\mathbf{M}^1)}\sum_{w\in W_{M,M}^{\mathrm{reg}}}\frac{|W^M|}{|W|}\frac{1}{|\det(w-1|\mathfrak{a}_M^G)|}\mathrm{tr}(M_{P|P}(w,0)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda,f)).$$

Since $\mathrm{tr}(M_{P|P}(w,0)\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda,f))$ is an invariant distribution and $\mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda)$ is admissible, we find that this is a finite linear combination of characters:

$$\sum_{\pi\in\Pi(\mathbf{G}^1)} a_{\mathrm{disc}}^G(\pi,\mathfrak{X})\mathrm{tr}\pi(f).$$

Here we note that $a_{\mathrm{disc}}^G(\pi) := \sum_{\mathfrak{X}\in\mathfrak{X}(G)} a_{\mathrm{disc}}^G(\pi,\mathfrak{X})$ are merely some scalars and are not the multiplicity of π in the discrete spectrum (cf. Th. 2.4)

$$L_{\mathrm{disc}}^2(G(F)\mathfrak{A}_G\backslash\mathbf{G}) = \bigoplus_{[G,\pi]} L^2(G(F)\mathfrak{A}_G\backslash\mathbf{G})_{[\pi]}.$$

We now give an expression analogous to Th. 4.5 for the sum $J(f) = \sum_{\mathfrak{X}\in\mathfrak{X}(G)} J_{\mathfrak{X}}(f)$. Set

$$\Pi_{\mathrm{disc}}(M) := \left\{ \begin{array}{r} \pi \in JH(\mathcal{I}_Q^M(\sigma_\mu)) \\ Q \in \mathcal{P}^M(L) \end{array} \middle| \begin{array}{l} (1)\ \ a_{\mathrm{disc}}^L(\sigma) \neq 0, \\ (2)\ \ w(\sigma_\mu) = \sigma_\mu,\ \exists w \in W_{L,L}^{M,\mathrm{reg}} \end{array} \right\},$$

$$\Pi(M_1) := \coprod_{M\in\mathcal{L}^{M_1}(M_0)} \{\pi_\lambda \mid \pi \in \Pi_{\mathrm{disc}}(M),\ \lambda \in i(\mathfrak{a}_M^{M_1})^*\}.$$

Note again that $\Pi_{\mathrm{disc}}(M)$ is not the set of discrete automorphic representations of M. For $\pi_\lambda \in \Pi(M_1)$, $\pi \in JH(\mathcal{I}_Q^M(\sigma_\mu))$ being as above, define

$$a^{M_1}(\pi_\lambda) := a_{\mathrm{disc}}^L(\sigma) r_L^{M_1}(\pi_\lambda), \quad r_L^{M_1}(\pi_\lambda) := \lim_{\Lambda\to 0} \sum_{\substack{Q\in\mathcal{F}(L) \\ Q\subset P_1}} \frac{r_Q(Q_1,\pi,\lambda,\Lambda)}{\theta_Q^{P_1}(\Lambda)}.$$

Here $r_L^{M_1}(\pi_\lambda)$ is independent of Q_1 and P_1.

Corollary 5.10 ([12]). *For $f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$, we have*

$$J(f) = \sum_{M\in\mathcal{L}(M_0)} \frac{|W^M|}{|W|} \int_{\Pi(M)} a^M(\pi)\mathcal{J}_M(\pi,f)\,d\pi.$$

Here we have written $\mathcal{J}_M(\pi,f) := \mathcal{J}_M(\pi,0,f)$ and the measure $d\pi$ on $\Pi(M)$ is such that

$$\int_{\Pi(M)} \phi(\pi)\,d\pi = \sum_{L\in\mathcal{L}^M(M_0)} \frac{|W^L|}{|W^M|} \sum_{\pi\in\Pi_{\mathrm{disc}}(L)} \int_{i(\mathfrak{a}_L^M)^*} \phi(\pi_\lambda)\,d\lambda$$

holds.

Proof. Write R_{disc}^M for the right regular representation of $\mathbf{M}$ on the discrete spectrum $L_{\mathrm{disc}}^2(M(F)\mathfrak{A}_M\backslash\mathbf{M})$ of M. We may consider the induced representation $\mathcal{I}_P^G(R_{\mathrm{disc},\lambda}^M)$, which from Th. 2.4 is isomorphic to $\bigoplus_{\mathfrak{X}\in\mathfrak{X}(G)} \bigoplus_{\pi\in\Pi(\mathbf{M}^1)} \mathcal{I}_{P,\mathfrak{X}}^G(\pi_\lambda)$. Th. 5.9 asserts that

$$\begin{aligned} J(f) = &\sum_{L\in\mathcal{L}(M_0)} \sum_{M\in\mathcal{L}^L(M_0)} \sum_{w\in W_{M,M}^{L,\mathrm{reg}}} \frac{|W^M|}{|W|} \frac{1}{|\det(w-1|\mathfrak{a}_M^L)|} \\ &\times \frac{1}{|\mathcal{P}(M)|} \sum_{P\in\mathcal{P}(M)} \int_{i(\mathfrak{a}_L^G)^*} \mathrm{tr}\big(\mathcal{M}_L(P,\lambda,0)M_{P|P}(w,0)\mathcal{I}_P^G(R_{\mathrm{disc},\lambda}^M,f)\big)\,d\lambda. \end{aligned}$$

Here $\mathcal{M}_L(P,\lambda,0)$, $M_{P|P}(w,0)$ are defined similarly as $\mathcal{M}_L(P,\pi,\lambda,0)$, $M_{P|P}(w,\pi)$ with π replaced with R_{disc}^M.

One can easily see that the operator $\mathcal{M}_L(P,\lambda,0)M_{P|P}(w,0)\mathcal{I}_P^G(R_{\mathrm{disc},\lambda}^M,f)$ vanishes on the orthogonal complement of a subspace which is a direct sum of $\mathcal{I}_Q^G(\pi_\lambda)$, ($Q \in \mathcal{P}(L)$, $\pi \in \Pi_{\mathrm{disc}}(L)$). On the $\mathcal{I}_Q^G(\pi_\lambda)$-component, $\mathcal{M}_L(P,\lambda,0)$ equals

$$\mathcal{M}_L(P,\pi,\lambda,0) = \sum_{Q\in\mathcal{P}(L)} \frac{1}{\theta_Q^G(\Lambda)} r_Q(P,\pi,\lambda,\Lambda)\mathcal{N}_Q(P,\pi,\lambda,\Lambda)\Big|_{\Lambda=0}.$$

Noting that $r_L^{P_1}(P, \pi, \lambda, \Lambda)$ (cf. Lem. 4.3) depends only on M_1 and not on $P_1 \in \mathcal{P}(M_1)$, we can write this as

$$\sum_{M_1 \in \mathcal{L}(L)} r_L^{M_1}(P, \pi, \lambda, \Lambda)\mathcal{N}_{M_1}(P, \pi, \lambda, \Lambda)\Big|_{\Lambda=0} = r_L^{M_1}(\pi_\lambda)\mathcal{N}_{M_1}(P, \pi, \lambda).$$

This combined with the definition of $a^L_{\mathrm{disc}}(\sigma)$ yields

$$J(f) = \sum_{L \in \mathcal{L}(M_0)} \sum_{M_1 \in \mathcal{L}(L)} \frac{|W^L|}{|W|} \sum_{\pi \in \Pi_{\mathrm{disc}}(L)} \int_{i(\mathfrak{a}_L^G)^*} a^L_{\mathrm{disc}}(\pi) r_L^{M_1}(\pi_\lambda) \mathrm{tr}\big(\mathcal{N}_{M_1}(Q, \pi, \lambda)\mathcal{I}_Q^G(\pi_\lambda, f)\big)\, d\lambda$$

decomposing $i(\mathfrak{a}_L^G)^* = i(\mathfrak{a}_L^{M_1})^* \oplus i(\mathfrak{a}_{M_1}^G)^*$,

$$= \sum_{M_1 \in \mathcal{L}(M_0)} \sum_{L \in \mathcal{L}^{M_1}(M_0)} \frac{|W^L|}{|W|} \sum_{\pi \in \Pi_{\mathrm{disc}}(L)} \int_{i(\mathfrak{a}_L^{M_1})^*} a^L_{\mathrm{disc}}(\pi) r_L^{M_1}(\pi_\lambda) \mathcal{J}_{M_1}(\pi_\lambda, f)\, d\lambda$$

writing M for M_1 and using the definition of the measure $d\pi$,

$$= \sum_{M \in \mathcal{L}(M_0)} \frac{|W^M|}{|W|} \int_{\Pi(M)} a^M(\pi) \mathcal{J}_M(\pi, f)\, d\pi.$$

□

6 The invariant trace formula

Recall, in many application of the trace formula, we need to compare the trace formulae of different groups. The starting point of the comparison is a correspondence between the conjugacy classes of the relevant groups. Consequently, we need to express the trace formula in terms of *invariant* distributions, distributions which are invariant under the conjugation. Here we shall explain how Arthur achieved this [4], [11], [12].

6.1 Non-invariance

First we measure the non-invariance of the terms in the trace formula. Recall the geometric kernel $K_Q^f(x, y) = \sum_{\mathfrak{o} \in \mathfrak{O}(G)} K_{Q,\mathfrak{o}}^f(x, y)$ of the induced operator $R_Q(f)$ on $L^2(\mathbf{V}L(F)\mathfrak{A}_G\backslash\mathbf{G})$ § 3.1. Since $K_{Q,\mathfrak{o}}^{\mathrm{Ad}(y^{-1})f}(x, x) = K_{Q,\mathfrak{o}}^f(xy^{-1}, xy^{-1})$, we have

$$\mathrm{Ad}(y)J_{\mathfrak{o}}^T(f) = \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} \sum_{Q \in \mathcal{F}(P_0)} (-1)^{a_L^G} \sum_{\delta \in Q(F)\backslash G(F)} K_{Q,\mathfrak{o}}(\delta x, \delta x)\widehat{\tau}_Q(H_Q(\delta xy) - T)\, dx$$

using (4.1),

$$= \int_{G(F)\mathfrak{A}_G\backslash\mathbf{G}} \sum_{Q\subset P\in\mathcal{F}(P_0)} (-1)^{a_L^M} \sum_{\delta\in Q(F)\backslash G(F)} K_{Q,\mathfrak{o}}(\delta x,\delta x)\widehat{\tau}_{Q^M}(H_Q(\delta x)-T)$$
$$\times \Gamma_P^G(H_Q(\delta x)-T,-H_Q(k_Q(\delta x)y))\,dx$$

$$= \sum_{P\in\mathcal{F}(P_0)} \int_{P(F)\mathfrak{A}_G\backslash\mathbf{G}} \sum_{\substack{Q\in\mathcal{F}(P_0)\\ Q\subset P}} (-1)^{a_L^M} \sum_{\delta\in Q(F)\backslash P(F)} K_{Q,\mathfrak{o}}(\delta x,\delta x)\widehat{\tau}_{Q^M}(H_Q(\delta x)-T)$$
$$\times \Gamma_P^G(H_P(\delta x)-T,-H_P(k_Q(\delta x)y))\,dx.$$

Here we have written $\delta x = q(\delta x)k_Q(\delta x)$ for the Iwasawa decomposition of δx with respect to $\mathbf{G}=\mathbf{QK}$. If we write $x\in P(F)\mathfrak{A}_G\backslash\mathbf{G}$ as $x=umak$, $(u\in\mathbf{U},\ m\in\mathbf{M},\ a\in\mathfrak{A}_M^G,\ k\in\mathbf{K})$, then

$$K_{Q,\mathfrak{o}}^f(\delta x,\delta x) = K_{Q^M,\mathfrak{o}^M}^{f_P^k}(\delta ma,\delta ma),\quad H_{Q^M}(\delta x)=H_{Q^M}(\delta m),$$
$$\Gamma_P^G(H_P(\delta x)-T,-H_P(k_Q(\delta x)y)) = \Gamma_P^G(H_P(a)-T,-H_P(ky)),$$

where

$$f_P^k(m) := m^{\rho_P}\int_{\mathbf{U}} f(k^{-1}muk)\,du,\quad m\in\mathbf{M}.$$

Using the function $\overline{f}_{Q,g}$ (see p. 24), we obtain the following. (The proof for the spectral formula is similar.)

Lemma 6.1 ([4] Th. 3.2). *For $f\in\mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$, we have*

$$\mathrm{Ad}(y)J_{\mathfrak{o}}(f) = \sum_{Q\in\mathcal{F}(M_0)} \frac{|W^L|}{|W|}J_{\mathfrak{o}}^L(\overline{f}_{Q,y}),$$
$$\mathrm{Ad}(y)J_{\mathfrak{X}}(f) = \sum_{Q\in\mathcal{F}(M_0)} \frac{|W^L|}{|W|}J_{\mathfrak{X}}^L(\overline{f}_{Q,y}).$$

We now explain the rough idea of the combinatorial part of the construction. Suppose we are given a family of continuous linear maps $\phi_L^M : \mathcal{H}(\mathbf{M})\to\mathcal{I}(\mathbf{L})$ satisfying

(1) $\phi_L^M(\mathrm{Ad}(y^{-1})f) = \sum_{P_1\in\mathcal{F}^M(L)} \phi_L^{M_1}(\overline{f}_{P_1,y})$,

(2) $\phi_M^M : \mathcal{H}(\mathbf{M})\to\mathcal{I}(\mathbf{M})$ is surjective,

(3) Any ($\mathrm{Ad}(\mathbf{M})$-) invariant distribution I^M on $\mathcal{H}(\mathbf{M})$ passes through ϕ_M^M: $I^M = \widehat{I}^M\circ\phi_M^M$.

Then we define a family of distributions $\{I_{\mathfrak{o}}^M, I_{\mathfrak{X}}^M\}_{M\in\mathcal{L}(M_0)}$ by

$$I_{\mathfrak{o}}^M(f^M) := J_{\mathfrak{o}}^M(f^M) - \sum_{L\in\mathcal{L}^M(M_0),\neq M} \frac{|W^L|}{|W^M|}\widehat{I}_{\mathfrak{o}}^L(\phi_L^M(f^M)),$$
$$I_{\mathfrak{X}}^M(f^M) := J_{\mathfrak{X}}^M(f^M) - \sum_{L\in\mathcal{L}^M(M_0),\neq M} \frac{|W^L|}{|W^M|}\widehat{I}_{\mathfrak{X}}^L(\phi_L^M(f^M)).$$

Then it follows from Lem. 6.1 and (1) that ($\bullet$ is $\mathfrak{o}$ or $\mathfrak{X}$)

$$\begin{aligned}
I_\bullet(\mathrm{Ad}(y^{-1})f - f) =& J_\bullet(\mathrm{Ad}(y^{-1})f - f) - \sum_{L\in\mathcal{L}(M_0),\neq G} \frac{|W^L|}{|W|}\widehat{I}_\bullet^L(\phi_L^G(\mathrm{Ad}(y^{-1})f - f))\\
=& \sum_{P\in\mathcal{F}(M_0),\neq G} \frac{|W^M|}{|W|} J_\bullet^M(\overline{f}_{P,y}) - \sum_{L\in\mathcal{L}(M_0),\neq G}\frac{|W^L|}{|W|}\sum_{P\in\mathcal{F}(L)} \widehat{I}_\bullet^L(\phi_L^M(\overline{f}_{P,y}))\\
=& \sum_{P\in\mathcal{F}(M_0),\neq G} \frac{|W^M|}{|W|}\left(J_\bullet^M(\overline{f}_{P,y}) - \sum_{L\in\mathcal{L}^M(M_0)} \frac{|W^L|}{|W^M|}\widehat{I}_\bullet^L(\phi_L^M(\overline{f}_{P,y}))\right)\\
=&0.
\end{aligned}$$

That is, $I_\bullet^M$ are all invariant distributions. On the other hand we deduce from (3.7) the *invariant trace formula*

$$\sum_{\mathfrak{o}\in\mathfrak{O}(G)} I_{\mathfrak{o}}(f) = \sum_{\mathfrak{X}\in\mathfrak{X}(G)} I_{\mathfrak{X}}(f).$$

6.2 Application of the trace Paley-Wiener theorem

Of course the most difficult point is to construct the maps ϕ_L^M. Arthur used the distribution $\mathcal{J}_M(\pi, X, f)$ (§ 5.4) for this. In fact, he defined the "Fourier transform" $\phi_M^G(f)$ of $f \in \mathcal{H}(\mathbf{G}_S)$ by

$$\phi_M^G(f) : \Pi_{\mathrm{temp}}(\mathbf{M}_S) \times \mathfrak{a}_{M,S} \ni (\pi, X) \longmapsto \mathcal{J}_M(\pi, X, f) \in \mathbb{C}.$$

Here $\Pi_{\mathrm{temp}}(\mathbf{M}_S)$ denotes the subset of tempered elements in $\Pi_{\mathrm{adm}}(\mathbf{M}_S)$. Of course this can be extended to $\Pi_{\mathrm{adm}}(\mathbf{M}_S) \times \mathfrak{a}_{M,S}$ by analytic continuation. It was shown in [13, I. Lem. 6.2] that this satisfies (1) above:

$$J_M(\pi, X, \mathrm{Ad}(y^{-1})f) = \sum_{Q\in\mathcal{F}(M)} J_M^L(\pi, X, \overline{f}_{Q,y}).$$

Here we overlook the technical imprecision that $\mathcal{H}(\mathbf{G}_S)$ is not stable under $\mathrm{Ad}(\mathbf{G}_S)$. The image $\mathcal{I}(\mathbf{G}_S)$ of $\mathcal{H}(\mathbf{G}_S)$ under ϕ_G^G is described by the trace Paley-Wiener theorem [19], [16].

The next problem is that the image $\mathcal{I}(\mathbf{M}_S)$ does not contain $\phi_M^G(\mathcal{H}(\mathbf{G}_S))$ if $G \neq M$. Then Arthur enlarged the range a little by relaxing the support condition in the direction of the center to obtain $\mathcal{I}_{\mathrm{ac}}(\mathbf{M}_S)$. To assure the surjectivity, he also enlarged the domain to a little larger space $\mathcal{H}_{\mathrm{ac}}(\mathbf{M}_S)$. Finally we have the surjective maps $\phi_L^M : \mathcal{H}_{\mathrm{ac}}(\mathbf{M}_S) \to \mathcal{I}_{\mathrm{ac}}(\mathbf{L}_S)$ for any $L \subset M \in \mathcal{L}(M_0)$.

The final problem is to show that the distributions $I_{\mathfrak{o}}^G$, $I_{\mathfrak{X}}^G$ pass through ϕ_G^G in order to define $\widehat{I}_{\mathfrak{o}}^G$, $\widehat{I}_{\mathfrak{X}}^G$. More precisely, we need to establish the following inductive statement.

Problem 6.2. *Suppose for any $M \in \mathcal{L}(M_0)$, $\neq G$, we are given the distributions $I_L^M(\gamma)$, $I_L^M(\pi)$ on $\mathcal{H}(\mathbf{M})$ which pass through ϕ_M^M, so that we can define $\widehat{I}_L^M(\gamma)$, $\widehat{I}_L^M(\pi)$, and satisfy*

$$\mathcal{J}_L^M(\pi, f^M) = I_L^M(\pi, f^M) + \sum_{M_1 \in \mathcal{L}^M(M_0), \neq M} \widehat{I}_L^{M_1}(\gamma, \phi_{M_1}^M(f^M)),$$

$$J_L^M(\gamma, f^M) = I_L^M(\gamma, f^M) + \sum_{M_1 \in \mathcal{L}^M(M_0), \neq M} \widehat{I}_L^{M_1}(\pi, \phi_{M_1}^M(f^M)).$$

Then the distributions

$$I_L^G(\gamma, f) := J_L^G(\gamma, f) - \sum_{M \in \mathcal{L}(M_0), \neq G} \widehat{I}_L^M(\gamma, \phi_M^G(f)),$$

$$I_L^G(\pi, f) := \mathcal{J}_L^G(\pi, f) - \sum_{M \in \mathcal{L}(M_0), \neq G} \widehat{I}_L^M(\pi, \phi_M^G(f))$$

pass through ϕ_G^G.

This was done at length in [11], [12]. We end this note by stating the resulting formula.

Theorem 6.3 (The invariant trace formula). *If we take a finite set of places S sufficiently large for $f \in \mathcal{H}(\mathbf{G}/\mathfrak{A}_G)$, then we have*

$$\sum_{M \in \mathcal{L}(M_0)} \frac{|W^M|}{|W|} \sum_{\gamma \in (M(F))_{M,S}} a^M(S, \gamma) I_M(\gamma, f)$$
$$= \sum_{M \in \mathcal{L}(M_0)} \frac{|W^M|}{|W|} \int_{\Pi(M)} a^M(\pi) I_M(\pi, f)\, d\pi$$

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